Reprinted from NUCLEAR SCIENCE AND ENGINEERING, Volume 15, No 1, January 1963
Academic Press Inc. *Printed in U S A.*

NUCLEAR SCIENCE AND ENGINEERING: **15**, 20–28 (1963)

Interaction of Fissionable Units*

HUGH K. CLARK

E. I. du Pont de Nemours & Co., Savannah River Laboratory, Aiken, South Carolina

Received March 28, 1962
Revised September 9, 1962

A simple technique is described for computing the interaction in groupings of fissionable units. The units may be slabs, cylinders, or spheres and may be surrounded by a reflector. The validity of the approximations employed is discussed. The technique is illustrated by applying it to the specific example of calculating the interaction in a reflected cubic array of eight spheres of uranium (93.5% U^{235}). Comparisons are made with experiment and with other methods of calculation.

I. INTRODUCTION

In ascertaining whether a proposed assemblage of fissionable units will be subcritical by a big enough margin to be considered safe, one of the more difficult and important problems is that of calculating the interaction within the assemblage. Although each unit, when isolated, may be subcritical by a substantial margin, the assemblage may be critical if the number of units is sufficiently large and if the units are sufficiently close to each other and to neutron-reflecting materials. Since nearly any material reflects substantial quantities of neutrons, it is necessary to include existing and possible additional reflection in any calculations of the actual interaction that may occur.

Because exact calculations for even the most idealized arrangements are extremely complicated, it is desirable to develop simple approximating procedures that are sufficiently accurate to permit safe arrangements to be calculated without requiring an excessively large margin of safety. The approach employed here focuses attention on the neutrons emitted by each unit and by the reflectors. Both an angular distribution of the neutrons emitted from an element of surface and a surface distribution of these sources are assumed, and the numbers of neutrons entering the units and the reflectors are computed. Dimensions, spacings, or compositions of the units are adjusted until the total neutron currents (integrated over the surfaces) entering the units have the values (relative to the total currents emitted) that are required to make k_{eff} unity for each unit and hence for the assemblage.

A brief description of the application of this approach to interactions between a single unit and a reflector and to interactions within an unreflected array is given in ref. *1*. A fuller description is given in ref. *2* in which extensive comparisons with experiments are made. The agreement with experiment is generally good, and the method has the desirable feature of usually predicting slightly more interaction than is found experimentally. For the sake of completeness some of these comparisons are included in the present paper. A brief description of the extension of the technique to reflected arrays is given in ref. *3*. A fairly complete description of the technique is given in ref. *4* in which the emphasis is on applications. The present paper sets forth and extends work that was summarized in ref. *3*.

A recent paper by Miraldi and Clark (*5*) employs a similar approach but has some features that make the calculations more complicated while actually resulting in a poorer approximation for the small, highly reactive units usually encountered in interaction problems. These authors include the transverse flux dependence while apparently disregarding the resulting outward bias in the angular distribution of emitted neutrons. They also include a term proportional to the square of the cosine of the angle that emitted neutrons make with respect to the normal to the emitting surface, in addition to

* The information contained in this article was developed during the course of work under contract AT(07-2)-1 with the U. S Atomic Energy Commission.

the term proportional to the first power. For fissionable units in which $| B^2/\Sigma^2 | \ll 1$, the latter part of their approach is valid, but for highly reactive units in which this is not so, it results in greater errors than the simpler approach employed in the present paper, in which only the first-power term is retained. A comparison of experiments with results calculated by means of the approximations in the present paper and with results calculated from the work of Miraldi and Clark is given in Table IV.

II. THEORY

A. Approximations

The flux (Φ) of monoenergetic neutrons in a uniform infinite medium in which scattering is isotropic satisfies (*6*) the wave equation

$$\nabla^2\Phi + B^2\Phi = 0 \tag{1}$$

in which B^2 (the material buckling) satisfies

$$\frac{B}{\Sigma} = c \tan^{-1}\frac{B}{\Sigma} \tag{2}$$

where c is the average number of secondary neutrons resulting from a neutron interaction with a nucleus, and Σ is the total cross section (or the transport cross section if the transport approximation is employed when scattering is anisotropic in the laboratory system). The above equations also hold exactly for polyenergetic neutrons in a hypothetical medium in which c and Σ are energy independent. The behavior of neutrons in an actual medium can be approximated closely by (1) and (2) with suitable average values for c and Σ.

The amplitude of the differential neutron current per unit solid angle passing through an element of area relative to the total current passing through the element is a function $F(\phi, \theta, \psi; x, y, z)$ of the angle ϕ that the differential current vector makes with the normal to the element, of the polar or azimuthal angle θ, of the angle ψ that the normal to the element makes with a reference direction, and of the coordinates of the element. If there is plane symmetry so that the only variation in Φ is in the x direction, if the positive x direction is taken to be the reference direction, and if $\psi = 0$, F is a function of only ϕ and x and can be shown (Appendix A) to be

$$F(\phi, x) = \frac{\cos\phi}{\pi} \frac{\Phi(x) - \dfrac{\cos\phi}{\Sigma}\dfrac{d\Phi}{dx}}{1 + (B/\Sigma\cos\phi)^2} \Bigg/ \left[\frac{\Phi(x)}{(B/\Sigma)^2}\log\left(1 + (B/\Sigma)^2\right) - \frac{2\Sigma}{B^2}\left(1 - \Sigma/B\tan^{-1}B/\Sigma\right)\frac{d\Phi}{dx}\right] \tag{3}$$

where $\Phi = A_1 \cos Bx + A_2 \sin Bx$. At this same element the total neutron currents in the positive and negative x directions can be shown to be, respectively

$$\begin{matrix} j^+ \\ j^- \end{matrix} = \frac{\log\left(1 + (B/\Sigma)^2\right)}{4B/\Sigma\tan^{-1}B/\Sigma}\Phi(x) \mp \frac{1 - \dfrac{\tan^{-1}B/\Sigma}{B/\Sigma}}{2B\tan^{-1}B/\Sigma}\frac{d\Phi}{dx} \tag{4}$$

In a finite medium, making the approximation that (1) and (2) are satisfied everywhere and that the current leaving the medium at a boundary (j^+ or j^-) is given by (4) requires that the current entering the region (j^- or j^+) must also be equated to (4). The effect of these boundary conditions on reactivity is equivalent to requiring continuity of neutron flux (Φ) and continuity of

$$\frac{B/\Sigma - \tan^{-1}B/\Sigma}{B\log\left(1 + (B/\Sigma)^2\right)}\frac{d\Phi}{dx}$$

For a wide range of B/Σ, $(1/3\Sigma)\,(d\Phi/dx)$ is a good approximation for this expression. The currents can therefore be represented by

$$\begin{matrix} j^+ \\ j^- \end{matrix} = \frac{\Phi}{4} \mp \frac{1}{6\Sigma}\frac{d\Phi}{dx} \tag{5}$$

Some indication of the validity of these approximations can be obtained by comparing the extrapolation distances (i.e., the extrapolated end points) obtained by setting j^- equal to zero with the exact values (*6*) (see Table I).

In the finite media with which we are concerned here, the assumptions will be made that (1) and (2) are satisfied everywhere and that (5) adequately represents the currents at boundaries. It will further be assumed that F is adequately represented by (3) for which a reasonable first order approximation is

TABLE I

Ratio of Approximate Extrapolation Distances to Exact Value (*6*)

B/Σ	Ratio by (4)	Ratio by (5)
0 5*i*	0 915	0 888
0	0.938	0.938
0 5	0.955	0 975
1.0	0 992	1.053
1 5	1 029	1 122
2.0	1.060	1.173

simply

$$F(\phi, x) = \frac{\cos \phi}{\pi} \tag{6}$$

In Fig. 1, $F(\phi, x)$ as given by (3) is plotted against solid angle for extreme values of x to indicate the extent of the approximation made by (6) when $B/\Sigma = 0$ and when $B/\Sigma = 1$. The distribution function employed by Miraldi and Clark (5) is

$$F(\phi, \theta; x, y, z) = \frac{\cos \phi}{\pi} \cdot \frac{\Phi(x, y, z) - \frac{1}{\Sigma}\left[\cos \phi \frac{\partial \Phi}{\partial x} - \sin \phi \cos \theta \frac{\partial \Phi}{\partial y} - \sin \phi \cos \theta \frac{\partial \Phi}{\partial z}\right]}{\Phi(x, y, z) - \frac{2}{3\Sigma}\frac{\partial \Phi}{\partial x}} \tag{7}$$

and is obtained by retaining only the first two terms in a Taylor expansion of the flux. At the center of symmetry of the (y, z) plane where $\partial\Phi/\partial y = \partial\Phi/\partial z = 0$, this function is identical with that obtained from (3) in the limit as $B/\Sigma \to 0$. For the small, highly reactive units usually encountered where interaction is important, however, B/Σ is not close to zero. For example, for the aqueous solution of U^{235} studied experimentally (7) with which they compare their calculations, $B_x/\Sigma \cong 0.5$ for small separations. Since their distribution function is represented by curves 1 and 2 of Fig. 1 regardless of the value of B/Σ, it is increasingly in error as B/Σ increases.

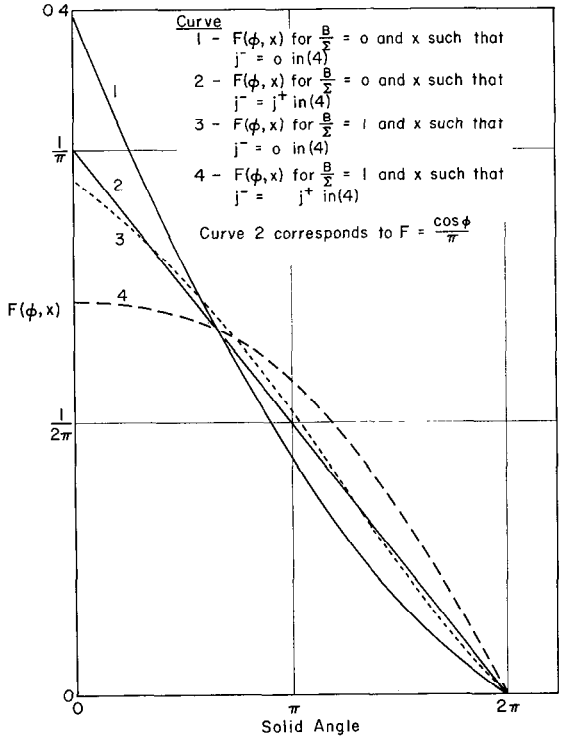

FIG 1. Angular distribution function $F(\phi, x)$ from (3) vs. solid angle $[2\pi(1 - \cos \phi)]$.

In employing (6) as the angular distribution function the dependence of F in θ, y, and z is ignored. The consistent approximation is to ignore the dependence of Φ on y and z in computing the fraction of the total current emitted from one surface that reaches another and, hence, to assume a uniform source strength per unit area on the emitting y, z surface. Miraldi and Clark include the y and z variation of the flux in their integrals, but they assert that, except for very small spacings, the contributions of the integrals containing $\partial\Phi/\partial y$ and $\partial\Phi/\partial z$ are small compared to those containing $\partial\Phi/\partial x$ and apparently ignore them since they report values only for the integrals containing $\partial\Phi/\partial x$. This assertion is no doubt true, but these contributions must be of the same order as the small differences between the integrals containing the y and z dependence of Φ and the much simpler integrals employed in the present paper.

Although the y and z variations in the flux are ignored in calculations involving F, geometric bucklings in the y and z directions are subtracted from the material buckling to obtain the buckling in the x direction. To avoid complicating the problem, boundary conditions are applied only in one dimension, i.e., to the y, z surfaces. If the arrangement of units is such that neutrons emitted from one slab can enter the x, y or x, z surfaces of another slab as well as the y, z surface, they are counted as entering the appropriate y, z surface. Boundary conditions are applied to the total currents obtained by integrating j^+ and j^- as given by (5) over the entire y, z surface. Where the surfaces are congruent the y and z variations of the flux cancel. Where the surfaces are not congruent and the y and z flux variations on one surface do not appreciably influence those on the other, the total currents are proportional to results given by (5) multiplied by the surface area. If the y and z variations on the two surfaces are not independent, as is the case when a large slab is close to a parallel small slab, the large slab should be considered to consist of a number of small slabs in edge-to-edge contact.

B. Calculation of Neutron Interchange

One of the chief advantages in approximating (3) by (6) is that, since F as given by (6) has no de-

pendence on Φ and $d\Phi/dx$, the interaction problem is readily divided into the geometrical problem of computing the fraction of the neutrons emitted from a given unit reaching another unit and the problem of calculating the critical thickness or buckling of a slab from boundary conditions expressed in terms of such fractions. Attention will be given first to the geometrical problem. The fraction ρ_{jk} of the total neutron current emitted from a surface A_k that enters another surface A_j is given by

$$\rho_{jk} = \iint \frac{\cos \phi_j \cos \phi_k \, dA_j \, dA_k}{\pi R^2 A_k} \tag{8}$$

where R is the distance between the elements of surface dA_j and dA_k and where ϕ_j and ϕ_k are the angles R makes with the normals to the elements. This integral is readily evaluated only in a few cases. For two parallel congruent planes having only one finite dimension and so arranged in space that a common normal joins equivalent points, the result is

$$\rho = \sqrt{1 + \alpha^2} - \alpha \tag{9}$$

where α is the ratio of the distance between the planes to the finite dimension. For two parallel discs similarly arranged but not necessarily congruent

$$\rho = \frac{1 + \gamma^2 + \alpha^2 - \sqrt{(1 - \gamma^2 - \alpha^2)^2 + 4\alpha^2}}{2} \tag{10}$$

where α is the ratio of the distance between the discs to the radius of the transmitter and γ is the ratio of the radius of the receiver to the radius of the transmitter. If two rectangular surfaces ($2a \times 2h$ and $2d \times 2g$) are oriented in perpendicular planes so that edges $2h$ and $2g$ are parallel (Fig. 2), the fraction (ρ) of the current emitted from $2a \times 2h$ that reaches $2d \times 2g$ is

$$\rho = \frac{1}{8\pi ah} \sum \left[G(z, z')H(x, y') \tan^{-1} \frac{G(z, z')}{H(x, y')} + \frac{[G(z, z')]^2}{4} \log \frac{[G(z, z')]^2 + [H(x, y')]^2}{[G(z, z')]^2} - \frac{[H(x, y')]^2}{4} \log \frac{[G(z, z')]^2 + [H(x, y')]^2}{[H(x, y')]^2} \right] \tag{11}$$

where $G(z, z') = z + f - z'$ and $[H(x, y')]^2 = (x + e)^2 + (y' + a + b)^2$ and where b is the distance from the plane of the $2d \times 2g$ rectangle to the closer $2h$ edge of the other rectangle, $d + e$ is the distance from the projection of the $2h$ edge on the plane of the $2d \times 2g$ rectangle to the further $2g$ edge, f is the distance between planes perpendicular to the $2g$ and $2h$ edges at their midpoints. The summation (with proper regard to sign) is over the 16 terms resulting from setting y' equal to $-a$ and a, x equal to $-e$ (or $-d$ if $e > d$) and d, z equal to $-g$ and g, and z' equal to $-h$ and h.

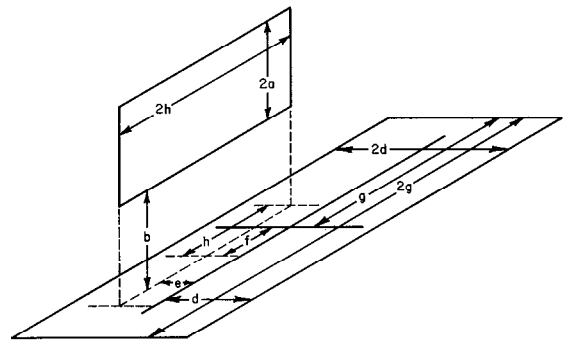

FIG 2. Interaction between perpendicular rectangles. (The $2g$ and $2h$ edges are parallel)

By noting that the numerator of (8) is the same whichever surface is the emitter and that the sum of all the fractions emitted from a surface equals unity, other interactions can be calculated. The fraction of the current emitted from the inner surface of an annulus that re-enters the annulus can be obtained from (10) as

$$\rho = 1 + \frac{\alpha}{2} \sqrt{1 + \frac{\alpha^2}{4}} \tag{12}$$

where α is the ratio of the height of the annulus to its inner radius. In refs. *2* and *4*, ρ for parallel rectangles is similarly obtained from (11) and is plotted against $\alpha (0 \leqq \alpha \leqq 1.4)$ and against $1/\alpha (1 \leqq \alpha \leqq \infty)$ for $\sigma = 0, 0.2, 0.4, 0.6, 0.8$, and 1.0 where α is the ratio of the separation to the shorter edge and σ is the ratio of shorter edge to the longer. The integrals given by Miraldi and Clark can be presented in this same convenient manner by multiplying (their notation) I_α by $\pi\epsilon\delta^2/16$ and I_β by $3\pi\epsilon\delta^3/32$.

C. Extension to Curved Surfaces

It has been tacitly assumed so far that the interacting surfaces are planes. Extension to curved surfaces can readily be made although the approximations are probably poorer than for planes. Equation (5) is replaced with its equivalent in spherical or cylindrical coordinates. For finite cylinders the material buckling is reduced by subtracting the axial buckling. The assumption of a uniform source strength per unit area might appear to be poor, but even for the extreme case of assuming the same source strength on both surfaces of each of a pair of interacting slabs, the assumption is not bad (see Section IV) provided the separation is not too small. (For slabs the assumption is not necessary to make the problem one-dimensional.)

Evaluation of (8) for curved surfaces is difficult. For a pair of infinite cylinders the result is identical with the Dancoff (8) correction when $\Sigma = 0$ in the medium surrounding the cylinder, and this calculation is available (9). On the basis of symmetry the fraction of the current from one cylinder reaching another must be at least as great as the fraction of 2π radians subtended at the axis of the emitter and for a pair of spheres at least as great as the fraction of 4π steradians subtended at the center of the emitter. The fraction is actually greater for a single closely spaced pair because in an array with the same spacing there is partial shielding. (Consider, for example, a cylinder surrounded by six others in contact with it.) On physical grounds, one would not expect the interaction in a group of circular cylinders to be altered appreciably by replacing each circular cylinder by a square cylinder of equal volume. Similarly it seems reasonable to replace spheres by cubes of equal volume. Such replacements maintain the same average density of material in an array without greatly altering the reactivity that an individual isolated unit would have. If, for any interaction between pairs, the square cylinders or cubes are assumed to rotate so that the interaction is only between a single pair of parallel rectangles, then the fraction of the current from one that reaches the other can be obtained by dividing the result for the parallel rectangles obtained from (11) by 4 for a cylinder or by 6 for spheres, since the orientation is such that only one face is involved. Results obtained in this way from (9) for infinite cylinders are compared in Table II with the Dancoff correction (9) and with the fraction of 2π radians subtended at the axis; results for spheres are compared in Table III with the fraction of 4π steradians subtended at the center. It is seen that the square cylinder and cube approximations give results on the high side for close spacings. These approximations, however, tend to compensate for assuming a constant source strength, whereas actually it is higher on facing surfaces; they give fairly good results when combined with other features of the present technique in making comparison with experiment (2).

TABLE II

FRACTION OF THE CURRENT FROM THE SURFACE OF AN INFINITE CYLINDER REACHING THE SURFACE OF A PARALLEL, IDENTICAL CYLINDER

(Radius)/(Axis-to-axis separation)	Fraction of 2π subtended at axis	Dancoff correction (9)	Square cylinder approximation
0	0	0	0
0 1	0 0319	0 0319	0 0266
0 2	0.0641	0.0646	0.0641
0.3	0.0970	0.0987	0.1130
0.4	0 1310	0.1360	0.1676
0 5	0.1667	0 1817	0.2199

TABLE III

FRACTION OF THE CURRENT FROM THE SURFACE OF A SPHERE REACHING THE SURFACE OF AN IDENTICAL SPHERE

(Radius)/(Center-to-center separation)	Fraction of 4π subtended at center	Cube approximation
0	0	0
0 1	0.0025	0.0020
0 2	0 0101	0.0103
0 3	0 0230	0 0305
0.4	0.0417	0.0641
0 5	0 0670	0 1072

D. SHIELDING

In any array of units there will be some shielding of more distant units by the closer units. Where the more distant units are completely blocked, $\rho_{jk} = 0$. Where the shielding is only partial, estimates are required. It is helpful to make use of the symmetry of the array, and in the case of finite cylinders it is helpful to extend them to infinity and to make use of the fact that the sum of the fractions emitted cannot exceed unity in determining the most distant cylinders that can be "seen." Where a unit in an array can "see" past surrounding units to a reflector surrounding the array, the fraction of the emitted current that reaches the reflector is easily obtained by subtracting from unity the fractions reaching other units. The fraction emitted by the reflector that reaches the unit can be obtained from the relation

$$\rho_{jk} = \frac{A_j}{A_k}\rho_{kj} \tag{13}$$

derived from (8), which holds provided a uniform source distribution over both surfaces may justifiably be assumed.

E. ARRAYS OF UNITS

Attention is now directed toward the second phase of the interaction problem. Consider a reflected array of fissionable units. Let the coordinates within each unit and within each reflector be chosen so that J^+ represents the total current emitted and J^- represents the total current received[1] where J^+ and J^- are

[1] This notation is slightly different from that used in refs. 1–4 where the emitted current for units is termed J_{in} and that for reflectors J_{out}.

equal to j^+ and j^- as given by (5) at the surface multiplied by the surface area under consideration (both y, z planes for slabs; curved surface for cylinders and spheres). Then for an assemblage of n units and reflectors

$$\begin{aligned} J_1^- &= \rho_{11}J_1^+ + \rho_{12}J_2^+ + \cdots + \rho_{1n}J_n^+ \\ J_2^- &= \rho_{21}J_1^+ + \rho_{22}J_2^+ + \cdots + \rho_{2n}J_n^+ \\ J_n^- &= \rho_{n1}J_1^+ + \rho_{n2}J_2^+ + \cdots + \rho_{nn}J_n^+ \end{aligned} \quad (14)$$

Each unit and reflector is numbered. Unless a surface is concave (e.g., the surface of reflector surrounding an array), $\rho_{jj} = 0$. When two units j and k are identical, J_j^+ differs from J_k^+ only in the constant factors appearing in the neutron current, but the number of equations is not reduced unless the units occupy symmetrically equivalent positions so that $J_j^+ = J_k^+$. If, for example, units 1 through m are symmetrically equivalent, (14) may be collapsed to

$$\begin{aligned} J_1^- &= \sum_{k=1}^{m} \rho_{1k}J_1^+ + \rho_{1,\,m+1}J_{m+1}^+ \\ &\qquad + \cdots + \rho_{1n}J_n^+ \\ J_{m+1}^- &= m\rho_{m+1,1}J_1^+ + \rho_{m+1,m+1}J_{m+1}^+ \\ &\qquad + \cdots + \rho_{m+1,n}J_n^+ \\ J_n^- &= m\rho_{n1}J_1^+ + \rho_{n,m+1}J_{m+1}^+ \\ &\qquad + \cdots + \rho_{nn}J_n^+ \end{aligned} \quad (15)$$

If more than one energy group of neutrons is employed, as may be desirable when a reflector causes appreciable changes in the neutron spectrum, the J's are vectors and the ρ_{jk}'s are scaler matrices. The symmetry conditions at the center of each unit or an outer boundary condition in a surrounding reflector make it possible to obtain expressions for Φ and $\nabla\Phi$ at the interacting surface that contain one undetermined constant per neutron group regardless of the number of concentric regions within a unit. Equation (14) can therefore be expressed as a matrix equation of the form

$$QC = PC \quad (16)$$

where C is a vector containing gn undetermined constants where g is the number of energy groups. A solution to (16) exists if and only if the determinant of $Q - P$ equals zero.

Although a solution can be obtained by adjusting spacings and dimensions of units to make the determinant zero (care being taken to get the physically meaningful root), it may be desirable to express the undetermined constants for the less reactive units and for the reflectors in terms of the constants for the most reactive unit and hence to reduce (16) to an equation involving only the constants of this unit. The determinant to be solved is thereby reduced to gr order where r is the number of classes into which symmetrically equivalent units may be grouped (e.g., in a $3 \times 3 \times 3$ cubic array $r = 4$: body center, face center, edge center, and corner). If this reduction is made, it may also be desirable to express J^- as βJ^+ where β is a $g \times g$ matrix so that in (16) Q is a function only of the dimensions and properties of the most reactive units and P is a function only of the other units and the ρ_{jk}'s.

In the above treatment as applied to slabs the flux within a unit is assumed to be symmetrical. For a pair of parallel slabs having congruent surfaces, the fraction (ρ_{ij}) of the neutrons emitted from one slab reaching the other, then, is one-half the fraction emitted from the facing surface. If one does not wish to make this assumption, (15) still has the same form, but the subscripts refer to the different slab surfaces. For surfaces that do not see reflectors or other surfaces, the incoming current is zero. When the flux is asymmetrical there are two undetermined constants per neutron energy group in the expressions for Φ and $\nabla\Phi$, but the same constants are involved in the equations for the two faces so that an equation of the form of (16) is still obtained.

F. One Group

For many purposes one energy group is sufficient (*1–4*). The reduction of the determinant to rth order is desirable here. The critical size of the most reactive unit can then be calculated by equating β as obtained from (5), or its equivalent in cylindrical or spherical coordinates, to the physically meaningful characteristic value of a matrix involving only the ρ_{jk}'s and the properties of the less reactive units and reflectors. Since β is the ratio of the current entering a unit to the current leaving a unit, its reciprocal is equal to the albedo of the unit and β itself is the albedo that must be provided by the surroundings in order to make the unit critical.[2]

When only one group of neutrons is employed, it is difficult to arrive at properly weighted average values of Σ. There may also be some difficulty in the choice of B for a unit and $\kappa = iB$ for a reflector. These difficulties can be obviated to a great extent if data on the critical sizes of bare and reflected units exist. From (5) β can be expressed as

$$\beta = \frac{1 + (2/3\Sigma)(\nabla\Phi/\Phi)}{1 - (2/3\Sigma)(\nabla\Phi/\Phi)} \quad (17)$$

[2] In ref. *1–4* β for a reflector is so defined that $J^+ = \beta J^-$; β for fissionable units is defined as above.

When $\beta = 0$, $x = (\pi/2B) - S_0$ for a slab, or $r = (2.405/B) - S_0$ for a cylinder or $(\pi/B) - S_0$ for a sphere, where S_0 is the bare extrapolation distance. S_0 can be determined from a bare critical experiment by equating the geometric buckling to a reasonable estimate of the material buckling; hence Σ can be expressed in terms of B and S_0. Since the albedo (β_k^{-1}) of a reflector (k) in contact with a unit (j) equals β_j, β_k^{-1} can be expressed in terms of B and S where S is the extrapolation distance when the reflector is present (reflector saving).

Although the reflector saving is not greatly dependent on the radius of curvature of the reflector, the same is not true of the albedo; hence it must be adjusted if the radius of curvature to be employed differs from that used in the experiment. This adjustment can readily be made in terms of albedos for a flat surface (reflector in contact with a slab) and for the curved surface in the experiment. The result cannot be put in closed form for cylinders, but for spheres it is

$$\beta(R) = \frac{\beta_\infty + (R_0/R)(\beta_0 - \beta_\infty)/(1 + \beta_0)}{1 - (R_0/R)(\beta_0 - \beta_\infty)/(1 + \beta_0)} \qquad (18)$$

where β_∞^{-1} is the albedo of the reflector when $R \to \infty$ and β_0^{-1} is the albedo at the experimental radius (R_0).

III NUMERICAL EXAMPLE

To illustrate the technique, calculations were made for $2 \times 2 \times 2$ cubic arrays of 20 kg U (93.5% U^{235}) spheres ($R = 6.335$ cm) having a density of 18.8 gm/cm^3. The arrays were enclosed within reflecting cubic shells equivalent to H_2O in their properties and having various dimensions. All eight spheres are symmetrically equivalent; hence by (15) and (16) we obtain

$$\beta_1 = 3\rho_{12} + 3\rho_{13} + \rho_{17} + \frac{8\rho_{19}\rho_{91}}{\beta_9 - \rho_{99}} \qquad (19)$$

where unit 9 is the reflector. There are three interactions with spheres separated by a single lattice spacing (s), three with spheres $\sqrt{2}s$ apart, and one with a sphere at $\sqrt{3}s$. To obtain the interactions between the spheres, the cube approximation was made with $\rho_{jk} = \rho_{kj}$ interpolated from Table III. The fraction of the neutrons emitted by a unit reaching the reflector is $\rho_{91} = 1 - 3\rho_{21} - 3\rho_{31} - \rho_{71}$. To obtain ρ_{19} use was made of (13). The fraction of the neutrons emitted by the reflector returning to it is $\rho_{99} = 1 - 8\rho_{19}$.

Multigroup calculations made with the cross sections in Reference (*10*) give $k = 2.30$ and $B^2 = 0.0837$ cm^{-2}. The experimentally determined bare critical mass of uranium of the density and enrichment being considered is 52.0 kg (*10, 11*); hence the bare critical radius is 8.71 cm and with $B^2 = 0.0837$ cm^{-2}, $S_0 = 2.15$ cm in agreement with the value calculated in Reference (*10*). In a similar manner S for a water-reflected sphere is found to be 4.1 cm for a sphere having a critical radius of 6.76 cm, derived from the experimentally determined water-reflected critical mass (*11*). The same values of S were assumed for a slab. For a slab (17) becomes

$$\beta = \frac{\sin B(S - S_0)}{\sin B(S + S_0)} \qquad (20)$$

and for a sphere it becomes

$$\beta = \frac{\dfrac{\pi - BS}{1 + (\pi - BS)\cot BS} - \dfrac{\pi - BS_0}{1 + (\pi - BS_0)\cot BS_0}}{\dfrac{\pi - BS}{1 + (\pi - BS)\cot BS} + \dfrac{\pi - BS_0}{1 + (\pi - BS_0)\cot BS_0}} \qquad (21)$$

where S is the reflector saving that must be provided by the surroundings for criticality to occur. With $S = 4.1$ cm, $\beta_\infty = 1.818$, and $\beta_0 = \beta(6.76) = 3.093$ where β_∞ and β_0 are the reciprocals of the results given by (20) and (21) since they are defined in terms of J^+ and J^- of the reflector. The cubic reflector was approximated by a spherical shell of equal volume in determining the value of R to be used in (18) to obtain $\beta_9 = \beta(R)$.

Equating β_1 in (19) to β in (21) resulted in a set of S as a function of lattice pitch and reflector size. Only for the critical conditions are these S values consistent with the actual sphere radius; otherwise they correspond to other radii that would be critical if the interaction somehow were to remain the same as calculated for the actual radius (6.335 cm). The effective multiplication constant for a sphere can be calculated as

$$k_{\text{eff}} = \frac{2.30}{1 + \dfrac{\pi^2}{(6.335 + S)^2}\dfrac{(2.30 - 1)}{0.0837}}$$

Values of k_{eff} for the array as a function of s and of the dimensions of the reflector are plotted in Figure 3. Safe spacings correspond to some choice of $k_{\text{eff}} < 1$.

The choice of a maximum safe value of k_{eff} for this or any other situation is necessarily somewhat arbitrary. On the one hand it should allow a sufficient margin to compensate for possible nonconservatism in calculations. Such a margin is best determined by comparing calculations with experiment. In any such comparison allowance must, of course,

TABLE IV

COMPARISON OF ρ_{12} AS DETERMINED BY EXPERIMENT AND AS CALCULATED BY VARIOUS PROCEDURES FOR A PAIR OF SLABS OF URANIUM SOLUTION

Separation (in)	Height (in.)	ρ_{12}				
		Experimental	Eq. (8)	Miraldi and Clark (*5*)	Tamor (*13*)	2 × Avg. solid angle (*14*)
2 25	10.0	0.693	0.770	0.851	0.711	0 598
6.25	12 9	0 468	0 569	0 658	0 440	0.393
15.25	17 7	0.266	0.362	0 442	0.188	0 218
20 25	19 8	0 212	0.304	0.362	0 124	0.179
30.25	23 5	0.141	0.216	0 259	0 055	0 121
48 25	28 8	0 083	0 134	0.167	0 015	0 073
66 25	32.3	0.055	0.091	0 112	0 004	0 051
∞	45 6	0	0	0	0	0

be made for experimental error. On the other hand the margin must be sufficient to allow for operational factors such as variations in unit size, composition, and spacing. The margin required to compensate for these factors requires studies of the administrative controls existing in the given situation and of the rate of change of k_{eff} with these factors.

An experimental result is available for a situation similar to the present example. By extrapolation of results obtained with up through 21 units a 3 × 3 × 3 cubic array of spheres with s = 11 in. was found (*12*) to have a reciprocal over-all multiplication of 0.031 when enclosed in a 3-ft cubic vault with concrete walls. Each sphere was equivalent in reactivity when isolated to one of the 20 kg uranium spheres being considered here. Calculations, reported in detail in ref. *4*, were made for an identical array of 20 kg uranium spheres enclosed in a 3-ft cubic shell of water, and k_{eff} was found to be 0.964. If, indeed, k_{eff} for the array studied experimentally is 1 − 0.031 = 0.969, the agreement between calculation and experiment would appear to be excellent. However, allowing for the smaller radius of the spheres in the experiment is estimated to reduce k_{eff} by as much as 5%. This reduction may be compensated for if concrete is an appreciably better reflector than water, but experiments with moderated systems indicate that water and ordinary concrete are nearly equivalent. In view of the uncertainty, it appears wise to pick 0.9 as the maximum safe value for k_{eff} to compensate for errors in the calculation. Additional margins required to compensate for variations in lattice pitch and in the size of the reflecting shell can be obtained from a study of Fig. 3.

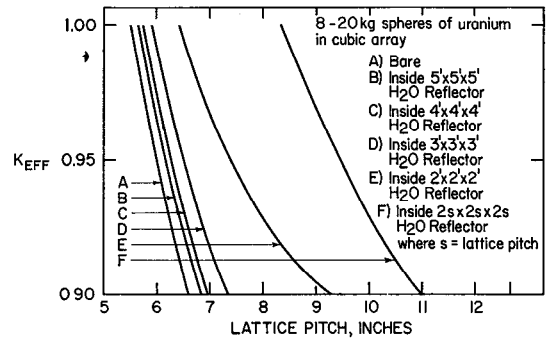

FIG. 3. k_{eff} as a function of lattice pitch

IV. OTHER COMPARISONS WITH EXPERIMENT

A large number of comparisons (*2*) have been made with experiments performed with aqueous solutions of U^{235}. The material bucklings were calculated from a simple one-group model and are no doubt somewhat in error, but the error cannot be excessively large since the extrapolation distances for bare and the water-reflected units, obtained by equating geometric bucklings to these material bucklings, are approximately equal to those obtained by more sophisticated methods (e.g. multigroup). For interacting units (S_0), obtained in this manner from data for an isolated bare unit, was employed on the noninteracting surfaces. An effective reflector saving (S) for the interacting surfaces was determined by equating geometric and material bucklings. Comparisons with experiment were made in terms of these experimental values of S and the values calculated by the technique being described here.

In one experiment (*7*), two 6-in.-thick, 47.5-in.-long slabs of solution were separated by various distances and the common critical height was determined. By working backwards from (20) with S_0 = 3.0 cm and B_m^2 = 0.02331 cm^{-2}, it is possible to obtain the value of ρ_{12} for the interaction between the two facing surfaces (not the entire units) that ap-

parently existed in the experiment.[3] In Table IV this experimental value of ρ is compared with values calculated from (8), with values calculated from the integrals tabulated by Miraldi and Clark, values calculated from formulas given by Tamor (*13*), and is also compared with twice the average solid angle. Tamor employs (7) and extends the integrals over y and z to infinity by analytic continuation of the flux. The solid angle is employed by others (*14*) in connection with a different technique, and twice its value is the limit approached by curve 4 of Fig. 1 as $B/\Sigma \to \infty$.

Finally, since a more common criterion for comparison of experiment and theory is in terms of the value of k_{eff} that is calculated for the experimental dimensions, some comment is in order as to the range of values to which the comparisons in ref. *2* correspond. A calculated $k_{eff} > 1$ means that the calculations overestimate the interaction and hence are conservative. For the two slabs just treated, $k_{eff} > 1$ for all cases and has a maximum value of 1.025 at the 6.25-in. spacing. Assuming that the flux within each slab is symmetrical reduces k_{eff}, but k_{eff} still exceeds unity except for the 2.25-in. spacing where it is 0.985. The only situations treated in ref. *2* where $k_{eff} < 1$ are large diameter cylinders and closely spaced cylinders within a cluster. In the former case the effect on k_{eff} of interaction between the curved surfaces is small, and the apparent nonconservatism in the calculation that gives a k_{eff} of 0.99 may actually be the result of experimental error. In the latter case, the nonconservatism should be expected as the result of assuming a uniform source distribution. The minimum k_{eff} obtained was 0.937 for a cluster of seven 6-in.-diameter cylinders in contact containing uranium solution where $H/U^{235} = 44.3$. In all the clusters studied, however, $k_{eff} > 1$ for axis-to-axis spacings that were 2 in., and in some cases only 1 in., greater than the cylinder diameters.

APPENDIX A

The neutron current (j^+) in the positive x direction is

$$j^+ = \frac{c\Sigma}{2} \int_0^1 \int_0^\infty \mu \Phi(x) e^{-\Sigma r}\, dr\, d\mu \qquad (A.1)$$

[3] The results are relatively insensitive to S_0 and B_m^2. Thus if $S_0 = 2.0$, $B_m^2 = 0.02798$ cm^{-2} to give the same bare critical size, and ρ_{12} at separations of 2.25, 6.25, and 30.25 in. equals, respectively, 0.697, 0.486, and 0.155.

where $\mu = \cos\phi$. Replacing $\Phi(x)$ by the general solution to (1),

$$\Phi(x) = A_1 \cos Bx + A_2 \sin Bx \qquad (A.2)$$

setting $x = x_1 + \mu r$, and performing the integrations show j^+ at x_1 to be

$$j^+ = \frac{c\Sigma^2}{4B^2} \Phi(x_1) \log\left(1 + (B/\Sigma)^2\right) - \frac{(c-1)\Sigma}{2B^2} \left(\frac{d\Phi}{dx}\right)_{x_1} \qquad (A.3)$$

Using (2) to eliminate c yields (4). Performing just the r integration in (A.1), dividing the result by (A.3), eliminating c with (2), and converting the integration over μ to an integration over solid angle (Ω) from 0 to 2π where $d\Omega = -2\pi\, d\mu$ give (3) for the integrand, namely the fraction of the neutron current (j^+) emitted per unit solid angle in a direction making an angle ϕ with the x axis.

REFERENCES

1. H. K Clark, *Trans Am. Nuclear Soc* **1** (1), 97–98 (1958)
2. H K. Clark, Interaction of subcritical components. Report DP-312 (1958).
3. H. K. Clark, *Trans. Am. Nuclear Soc.* **3** (1), 291–292 (1960).
4. H. K. Clark, "Handbook of Nuclear Safety," Chap. V. Report DP-532, pp. 501–523 (1961)
5. F. Miraldi and M. Clark, Jr , *Nuclear Sci. and Eng.* **11**, 246–255 (1961).
6. K. M. Case, F. de Hoffman, and G. Placzek, "Introduction to the Theory of Neutron Diffusion," Vol. I. Los Alamos Scientific Laboratory, New Mexico, 1953.
7. J. K. Fox and L. W Gilley, Report ORNL-2389, pp. 71–83 (1957).
8. S. M. Dancoff and M. Ginsburg, Report CP-2157 (1944).
9. I. Carlvik and B. Pershagen, Report AE-16 (1959).
10. G. E. Hansen, *Proc. 2nd U.N. Intern. Conf. Peaceful Uses Atomic Energy, Geneva, 1958* **12**, 84–88 (1958).
11. H. C. Paxton and G. A. Graves, Los Alamos Scientific Laboratory Report LA-1958 (1956) (declassified)
12. E. C. Mallary, H. C. Paxton, and R. H. White, Los Alamos Scientific Laboratory Report LA-1875 (1958) (declassified with deletions April 1958).
13. J. A. McLennan, Report AECD-3958 (1952) (declassified).
14. H. F. Henry, J. R. Knight, and C. W. Newlon, Report K-1309 (1956) (declassified).