REFERENCE 118

H. E. HANDLER, "MEASUREMENT OF MULTIPLICATION CONSTANT FOR SLIGHTLY ENRICHED HOMOGENEOUS UO_3-WATER MIXTURES AND MINIMUM ENRICHMENT FOR CRITICALITY," HANFORD ATOMIC PRODUCTS OPERATION REPORT HW-70310 (AUGUST 1961).

HW-70310
UC-46, Criticality Studies
(TID-4500, 17th Ed.)

MEASUREMENT OF MULTIPLICATION CONSTANT FOR SLIGHTLY ENRICHED HOMOGENEOUS UO_3-WATER MIXTURES AND MINIMUM ENRICHMENT FOR CRITICALITY

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August 21, 1961

HANFORD ATOMIC PRODUCTS OPERATION
RICHLAND, WASHINGTON

Work performed under Contract No. AT(45-1)-1350 between the Atomic Energy Commission and General Electric Company

Printed by/for the U. S. Atomic Energy Commission

Printed in USA. Price $2.00. Available from the
Office of Technical Services
Department of Commerce
Washington 25, D. C.

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ABSTRACT

The minimum U^{235} enrichment required for criticality in an infinite homogeneous system of uranium trioxide (UO_3) and water has been determined from measurements of k_∞ in the Hanford Physical Constants Testing Reactor, (PCTR). This is the enrichment for which $k_\infty \leq$ unity in an aqueous homogeneous system.

The experiments consisted of k_∞ measurements at uranium enrichments of 1.006, 1.070, and 1.159 weight per cent U^{235} for hydrogen-to-uranium atomic ratios in the range of 3.5 - 7.5.

The minimum enrichment required for criticality in an infinitely large system with a homogeneous UO_3-water mixture was found to be $1.034 \begin{smallmatrix}+0.010\\-0.009\end{smallmatrix}$ weight per cent U^{235}.

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INTRODUCTION

The neutron multiplying properties of mixtures of uranium trioxide (UO_3) and water are of interest from the viewpoint of nuclear safety because of their occurrence in both the preparation and processing of reactor fuels containing uranium.

The experiments reported here were undertaken to provide information for nuclear safety specifications, in that, the determination of the minimum U^{235} enrichment in UO_3-water mixtures for which a chain reaction can be sustained greatly reduces the problems of nuclear safety in situations where this enrichment is not exceeded.

The minimum enrichment required for criticality in UO_3-water mixtures was determined by measurements of the infinite multiplication constant (k_∞) in the Physical Constants Testing Reactor (PCTR). The enrichment for which the maximum value of k_∞ is equal to unity is then the minimum enrichment required for criticality.

The intent of this report is to describe in detail the experimental and theoretical work performed in the determination of the minimum U^{235} enrichment required for criticality in UO_3-water mixtures.

DISCUSSION OF THE PCTR EXPERIMENT

The PCTR was designed primarily to permit direct measurements of the excess multiplication factors ($k_\infty - 1$) for reactor lattices.(1) The reactor is a 7 × 7 × 7-foot graphite moderated assembly driven by highly enriched fuel. The central cavity is 2 × 2 × 3-foot and the sample section of the system is placed in this cavity for study. The sample section consists of a central test sample surrounded by a "buffer" region which is a layer of material identical to the sample. The purpose of the buffer material is to cause the neutron energy spectrum to come to equilibrium in the central test sample. This is accomplished by having a "thick" enough buffer region and also by adjusting the highly enriched driving fuel in such a way that the equilibrium spectrum is the spectrum that would exist in an infinite reactor made up of sample material.

In principal the PCTR measurements consist of comparing the behavior of the reactor with the central test cell in position and with a void in the same region (a void has $k_\infty = 1$ neutron going into the void must simply pass through it and come out again). For each case the reactor is made slightly supercritical by withdrawing the control rods; the reactor period is then measured. These period measurements, together with a knowledge of the neutron spectrum and the sensitivity of the reactor, are then related to k_∞ by the appropriate calculations. The theory behind these calculations presupposes that the neutron spectrum is identical with the spectrum that would exist in an infinite, just critical ($k_\infty = 1$) system of this material. If the spectrum differs from the ideal case, then the results point out the direction to be taken to obtain the correct condition.

In practice the measurements consist of the following steps: The system (buffer and core tank) is poisoned by the addition of a suitable neutron absorber by an amount expected to reduce k_∞ to unity. A period measurement is then made with the core tank in place and then with the void (Helium tank) in place. If the reactivity measurements of the core tank and void are not equal, some additional poison (usually placed heterogeneously on the outside of the core tank) is added and another period measurement is made. Gold foils (bare and cadmium covered) are irradiated at positions in the core and buffer and yield information on the neutron flux at these positions. If the initial guess as to the amount of poison in the buffer region was incorrect, the system must be repoisoned and the measurements repeated.

In many cases poison in the buffer region may be unnecessary where k_∞ of the system is very nearly equal to unity.

Analysis of the data was carried out by calculations resulting from the theoretical treatment shown in the Appendices.

EXPERIMENTAL DETAILS

Because of the lack of agreement in the theoretical predictions of the minimum enrichment required for criticality, the experiment was first done in minature. The original calculations gave enrichment values of 0. 9 and 1. 7 weight per cent U^{235} as the minimum enrichment. If the lower value was assumed to be correct and the higher value actually was correct, a criticality hazard would exist in loading the tanks. If the higher value was assumed to be correct and the lower value actually were correct, there would be insufficient material to perform the experiment at the correct enrichment. Thus the choice was made to first run the experiment in miniature. In the miniature system, accuracy had to be sacrificed since the buffer thickness was insufficient to give reliable results. A picture of the miniature system is shown in Figure 1. The purpose of the "miniature" experiment was to determine the value of the minimum enrichment required for criticality as between 0. 99 - 1. 15 weight per cent U^{235} and the optimum hydrogen to uranium (H/U) atomic ratio at approximately seven. These results are shown in Figure 2.

Batches of material were then processed for the larger full scale experiments. These were prepared with enrichments of 1. 006, 1. 070 and 1. 159 weight per cent U^{235}, with H/U atomic ratios in the range 3.5 - 7.5. In addition buffer tanks were prepared with "thicknesses" of 2, 4, and 6 inches to investigate the effect of buffer thickness on the experimental results. Also, the effect of the aluminum walls on the results were investigated.

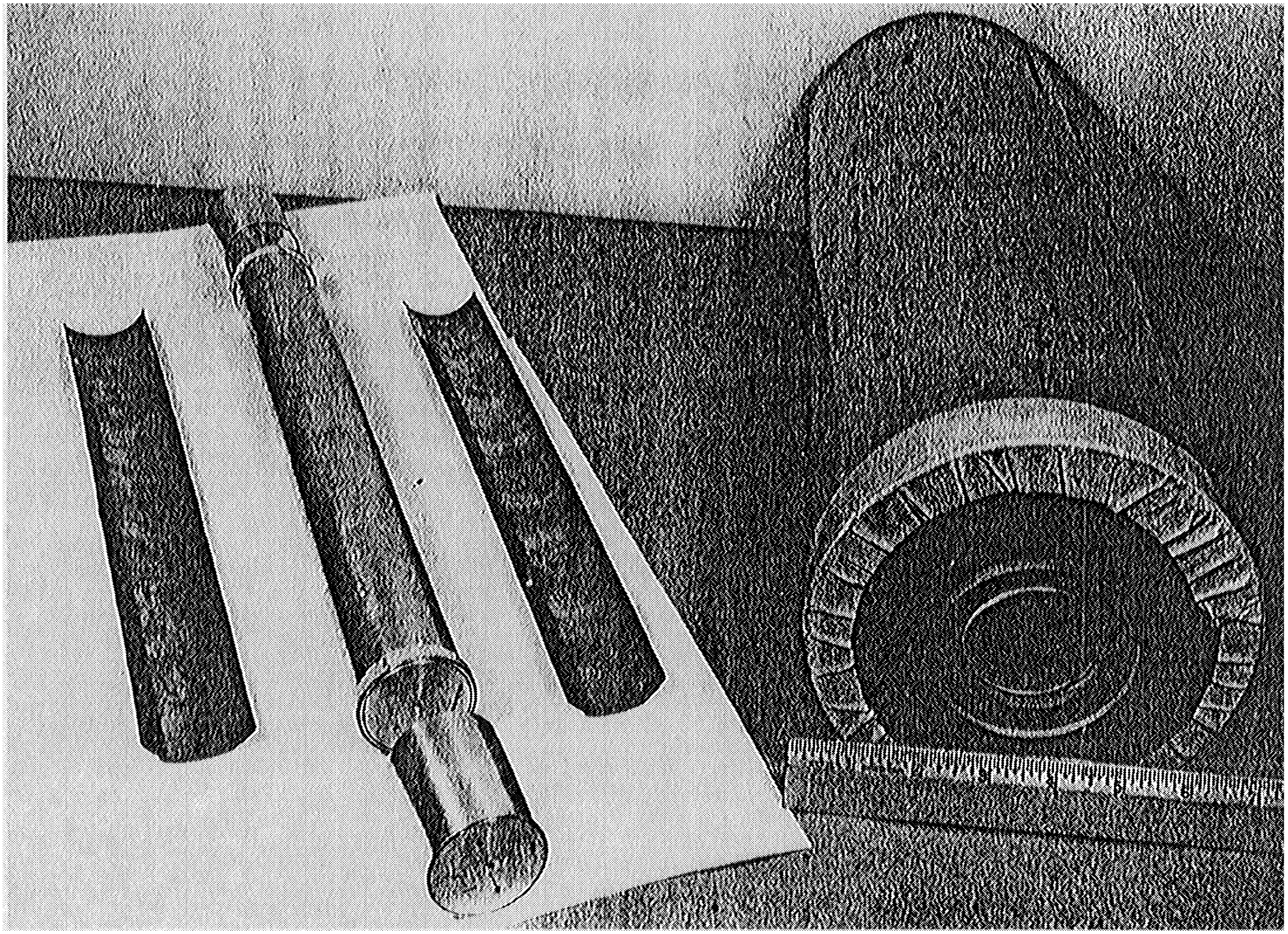


FIGURE 1

Miniature Tanks for Slightly Enriched UO_3 - H_2O k_∞ Experiments

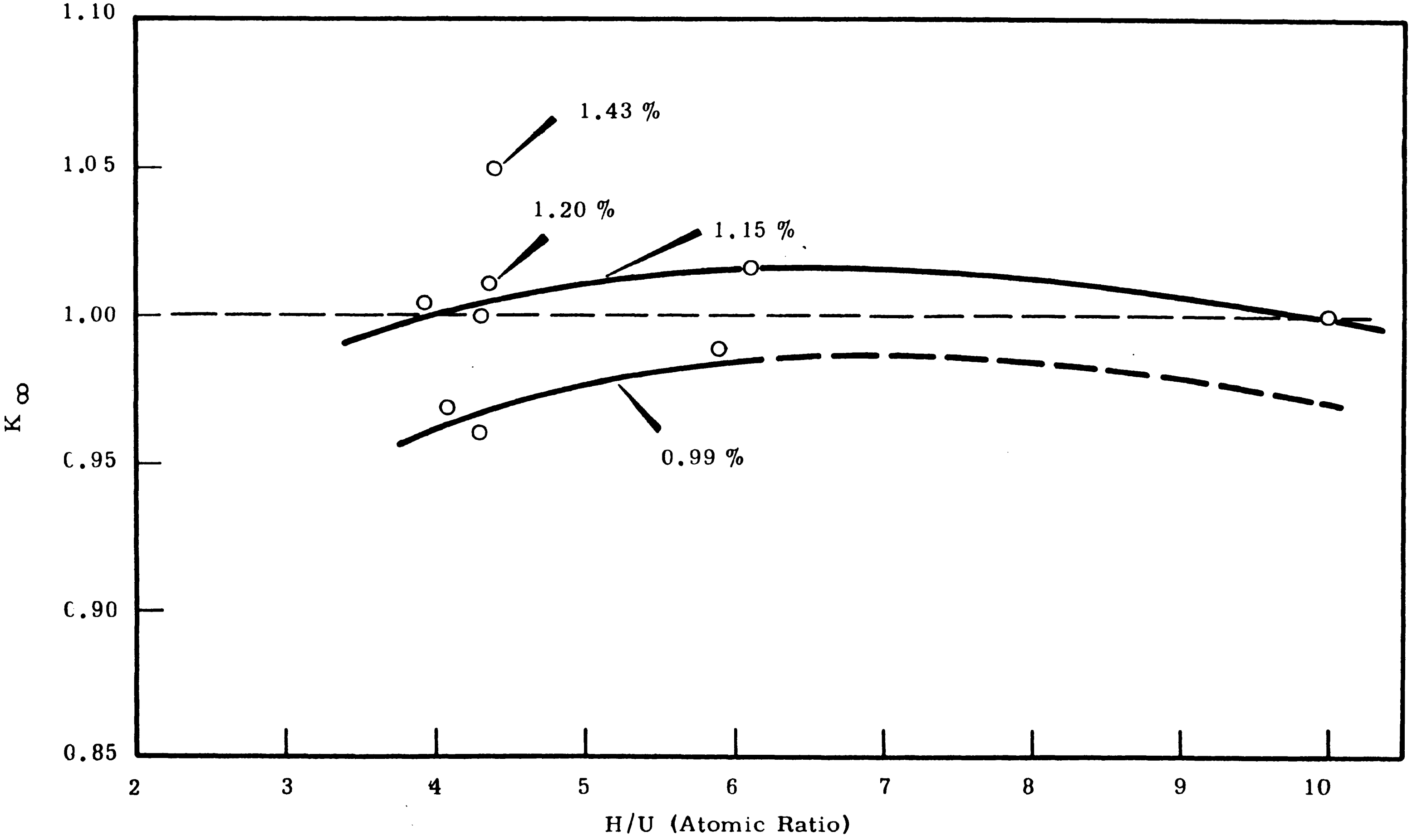


FIGURE 2

k_∞ of Enriched UO_3 - H_2O Mixtures: "Miniature" Experiment

The materials used in this experiment consisted of UO_3 powder at various states of water hydration and of absorbed water at enrichments of 1.006, 1.070 and 1.159 weight per cent U^{235}. The average particle size of the dry powder was 60 microns. Other materials in the system consisted of the 61 ST aluminum used in the containment vessels and strips of 10 mil copper used as a neutron absorber (poison) in the experiments. The preparation of these materials at the various enrichments and H/U ratios was performed by Chemical Research and Development Operation of Hanford Laboratories Operation. Analysis of the samples consisted of mass spectrometer determinations of the U^{235} weight per cent, determination of the weight per cent of water and uranium,and analysis for impurities such as nitrogen and other possible neutron poisons.

The containment vessels for these UO_3-water systems are shown in Figures 3 and 4. They consisted of an outside buffer tank which was an annular cylinder 36 inches long with an outside diameter of 18-1/2 inches and an inside diameter of 6-5/8 inches. This vessel had a 1/2-inch 61 ST aluminum outside wall and a 3/16-inch 61ST aluminum inside wall. One end plate was removable for loading purposes. A traverse hole, 1/2-inch in diameter and constructed from 61 ST aluminum tubing, was provided along the radius of the outside buffer tank for foil activation purposes. The inside buffer tanks were 8 inches long and 6-1/2 inches in diameter. Both the core and inside buffer tanks were constructed from 1/4-inch 61 ST aluminum. The inside buffer tanks were provided with traverse holes along the axis of the tank, and the core tank had traverse holes along the diameter and along the axis for foil activation purposes. Figure 5 is a schematic drawing of the tank assembly and supporting graphite in the PCTR cavity. Figure 6 shows the core tank being positioned in the assembled system in the PCTR cavity.

Because of the size and weight of these vessels, special equipment had to be assembled in order to move the vessels in and out of the PCTR cavity. A special cart was constructed and equipped with tracks and pneumatic lifting devices for handling the vessels; this cart can be seen in Figure 3.

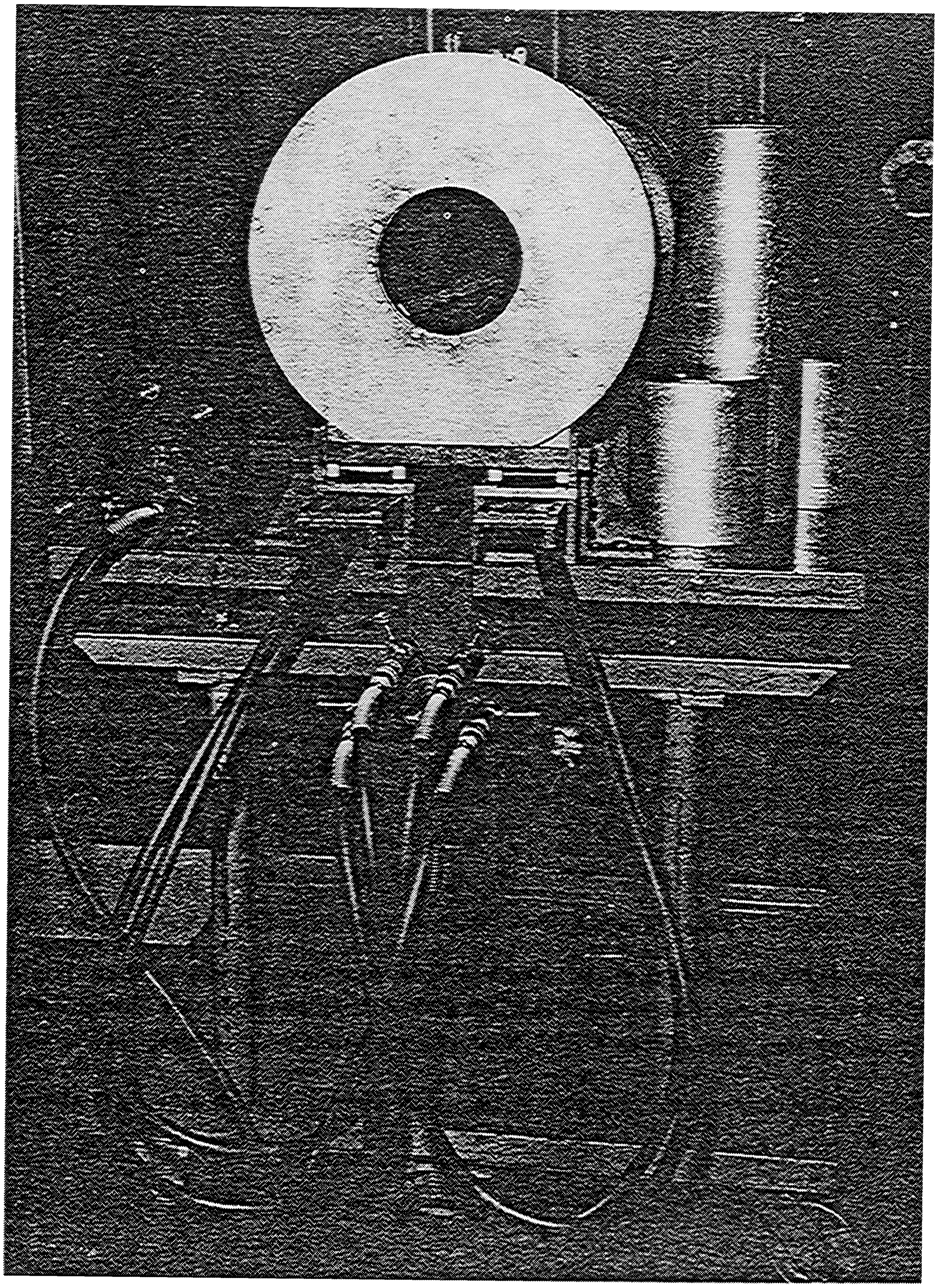


FIGURE 3

Containment Vessels for Slightly Enriched UO_3 - H_2O k_∞ Experiments

AEC-GE RICHLAND, WASH

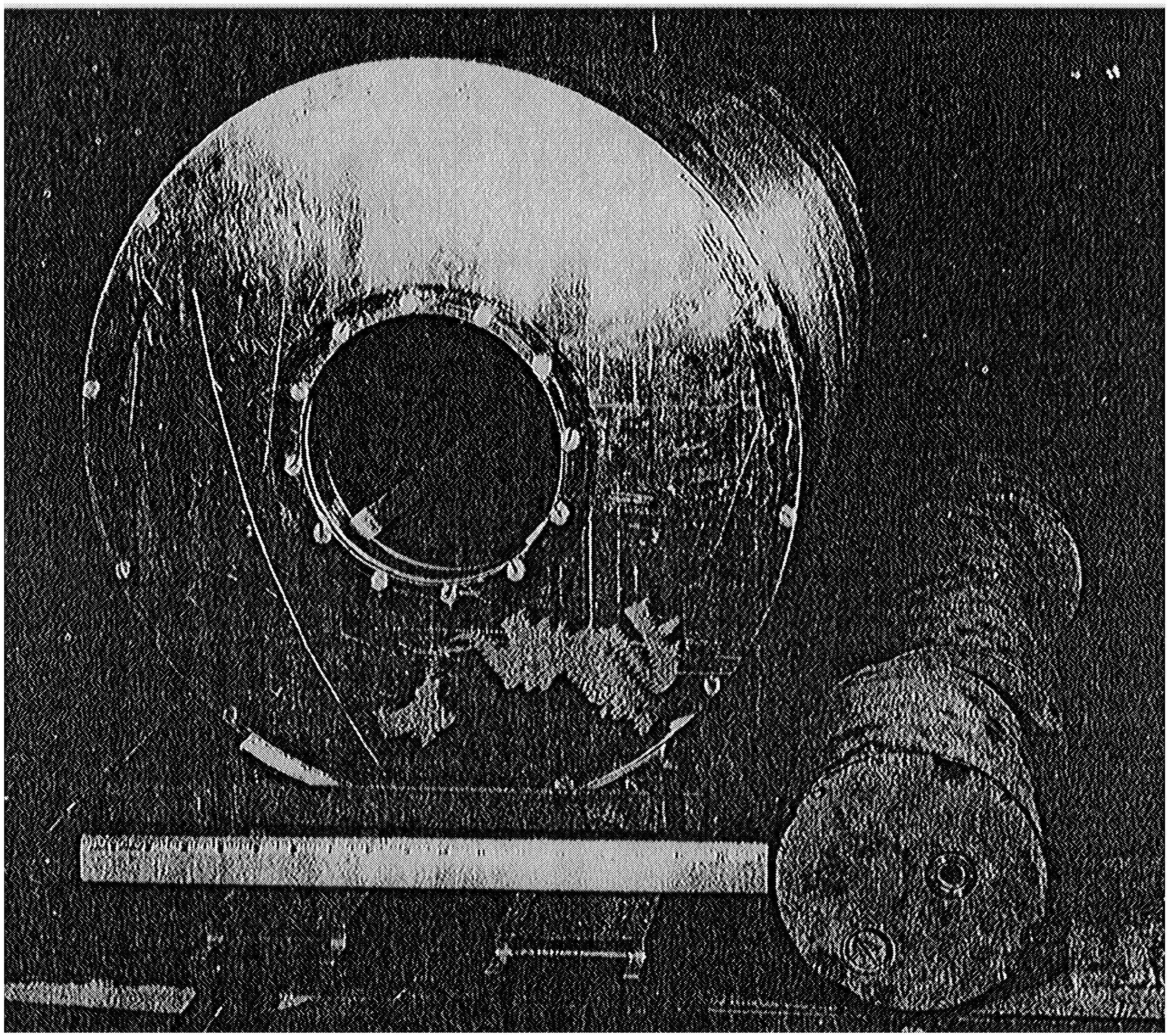
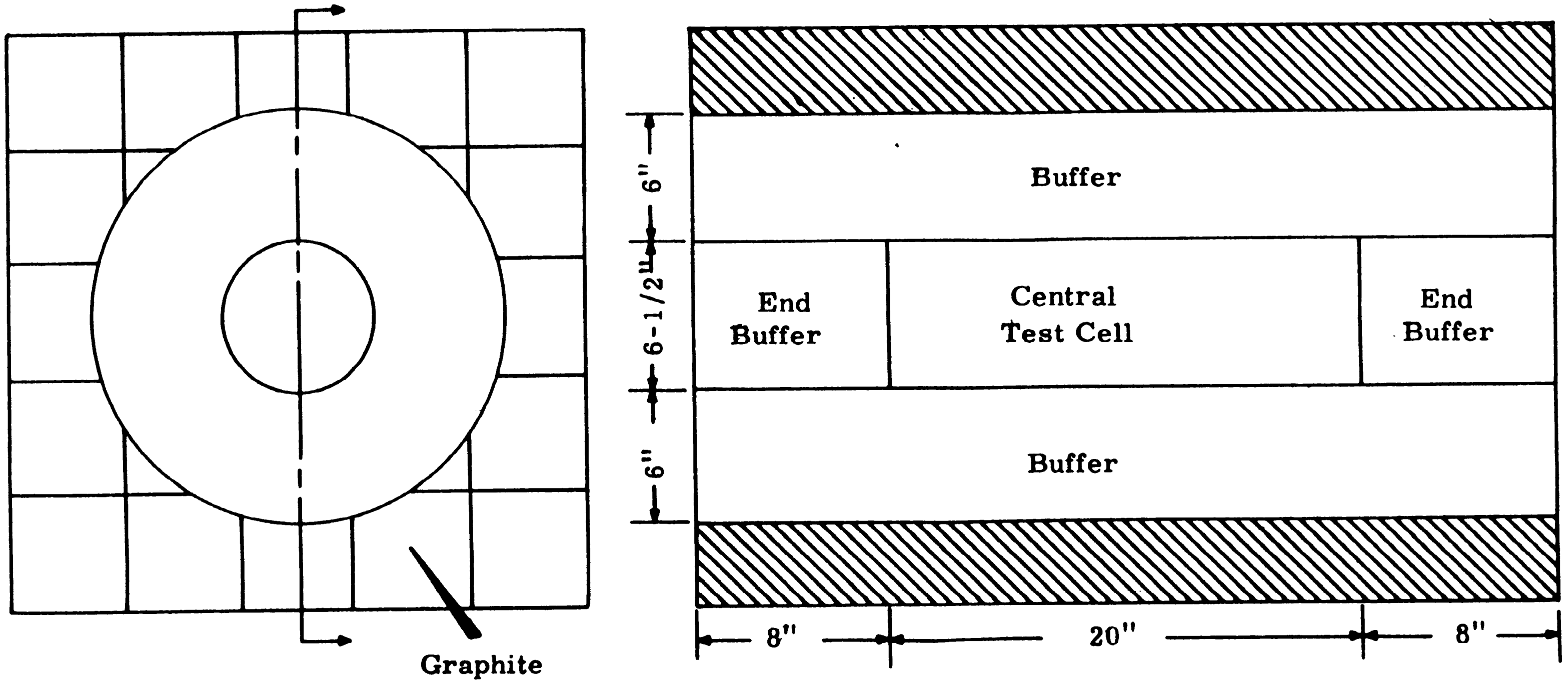


FIGURE 4

Containment Vessels for Slightly Enriched UO_3 - H_2O k_∞ Experiments



Front View

Side Section View

FIGURE 5

Typical Arrangement in PCTR Cavity with a Homogeneous System

FIGURE 6

Loading Core Tank into the Assembled System

DISCUSSION OF RESULTS

Data Analysis

The basic data for these experiments consists of the values from reactivity measurements, flux ratios from foil activations, U^{235} enrichment values from mass spectrometer analysis, water analyses; impurities analyses, and the weights of materials in the system.

The basic formula for calculating k_∞ is derived in the Appendices and only the result is repeated here. The basic equation and definitions are given as follows:

$$\Delta k_\infty' = \frac{\Delta P_{cv}}{\Delta P_{cp}} \; \frac{M_p}{M_c} \; \frac{\left(\frac{\sigma f}{A}\right)_p}{\left[\sum_i \left(\frac{\sigma f}{A}\right)_i \frac{Mi}{Mc}\right]} \; \frac{\bar{\phi}_p \, \phi_{pp}}{\bar{\phi}_c \, \bar{\phi}_v} \left(1 + 1.1\, pB\right)$$

ΔP_{cv}	is the reactivity difference between the unpoisoned and the helium tanks.
ΔP_{cp}	is the reactivity difference between the unpoisoned core and the poisoned core.
M_p	is the mass of poison.
M_c	is the mass of all materials in the core tank not including poison.
M_i	is the mass of the i^{th} material in the core tank not including poison.
σ	is the 2200 meter/sec microscopic cross section.
f	is the non-$\frac{1}{v}$ correction to the 2200 meter/sec cross section.
A	is the atomic or molecular weight.
$\bar{\phi}_p$	is the average "$\frac{1}{v}$" flux at the position of the poison before the poison is inserted.

$\bar{\phi}_c$ is the average "$\frac{1}{v}$" flux in the core

$\bar{\phi}_{pp}$ is the average "$\frac{1}{v}$" flux at the surface of the poison (copper).

$\bar{\phi}_v$ is the average "$\frac{1}{v}$" flux in the void.

p is the calculated resonance escape probability.

B is the correction due to resonance absorptions in the poison.

Now $\Delta k'_\infty$ is equal to $k'_\infty - 1$, where k'_∞ is the value of k_∞ uncorrected for possible flux mismatches between the buffer and core tanks, slight differences in the mass of aluminum in the various core tanks, possible poison effects from small amounts of impurities, and corrections due to the effect of the containment materials on the neutron energy spectrum.

Table I lists some of the basic data and the values of $\Delta k'_\infty$. The derivations and calculations of all of these parameters are contained in the Appendices.

The correct value of Δk_∞, that is, $k_\infty - 1$, is found by applying the above mentioned corrections to $\Delta k'_\infty$. These are as follows:

1. Flux Mismatch Between Core and Buffer

A number of k_∞ measurements were made for each core tank with the driving fuel in different configurations. From the foil activations of these various experiments, it was possible to determine the "correct" (no flux mismatch) configuration, and thus the correct value of k'_∞. The method of analysis of the foil activation measurements is discussed in the Appendices.

2. Aluminum Mass Difference Between Tanks

This correction was designated by the symbol δk_t and was simply to correct for any absorptions of neutrons in the aluminum caused by the differences in the masses of the aluminum tanks. The method is discussed in detail in the Appendices.

TABLE I

BASIC DATA AND VALUES OF $\Delta k'_\infty$ FOR SLIGHTLY ENRICHED HOMOGENEOUS UO_3-WATER MIXTURES

Material % U^{235}	H/U	Buffer Thickness In Inches	$M_p\left(\frac{\sigma f}{A}\right)_p$	$M_c\left[\sum_i\left(\frac{\sigma f}{A}\right)_i\frac{M_i}{M_c}\right]$	$\frac{\phi_p}{\phi_c}$	$\frac{\phi_{pp}}{\phi_V}$	$1 + 1.1\ pB$	$\frac{\Delta\rho_{cv}}{\Delta\rho_{cp}}$	$\Delta k'_\infty \times 10^3$
1.006	4	6	19.666	604.7	0.991	0.889	1.0224	-0.346	-10.14
			19.666		1.044	0.922	1.0238	-0.453	-14.52
	5	6	19.666	695.5	1.005	0.905	1.0206	-0.480	-12.61
			19.666		0.990	0.905	1.0217	-0.479	-12.41
			19.666		1.006	0.905	1.0213	-0.516	-13.57
			19.666		1.002	0.933	1.0207	+0.572	-15.43
	6	6	19.666		0.998	0.902	1.0181	-0.472	-12.54
			19.666		0.998	0.903	1.0184	-0.525	-13.96
			10.895		0.987	0.956	1.0181	-0.995	-15.36
			10.895		0.999	0.954	1.0183	-1.095	-17.06
	7	6	19.666	523.5	1.008	0.895	1.0160	-0.608	-20.94
			19.666		0.981	0.924	1.0182	-0.970	-33.63
	8	2	19.666	526.0	1.058	0.859	1.0136	-0.0295	-1.016
			19.666		1.030	0.897	1.0147	-0.4083	-14.31
	8	4	19.666	526.0	1.000	0.884	1.0148	-0.788	-26.42
			19.666		1.009	0.892	1.0156	-0.974	-33.27
			10.895		1.013	0.934	1.0146	-1.290	-25.63
			10.895		1.008	0.934	1.0154	-1.801	-35.67
	8	6	19.666	526.0	1.012	0.905	1.0157	-1.027	-35.70
			19.666		1.003	0.872	1.0162	-1.300	-43.21
			10.895		1.000	0.942	1.0156	-1.984	-39.31
			10.895		0.999	0.938	1.0160	-2.094	-41.26
1.070	4	6	19.666	559.5	1.022	0.916	1.0236	+0.312	+10.50
			19.666		0.978	0.923	1.0258	-0.0220	-0.717
	6	6	19.666	564.3	1.012	0.892	1.0201	+0.0836	+2.68
			19.666		1.017	0.917	1.0186	+0.374	+12.38
	8	6	19.666	581.3	1.019	0.877	1.0167	-0.119	-3.66
			19.666		1.000	0.904	1.0167	-0.382	-11.88
1.159	4	6	10.895	595.6	1.010	0.920	1.0261	+1.700	+29.64
			19.666		1.030	0.921	1.0242	+1.040	+33.36
	6	6	19.666	605.5	0.998	0.900	1.0199	+1.047	+31.15
			19.666		0.997	0.912	1.0208	+1.037	+31.26
			10.895		1.009	0.950	1.0197	+1.892	+33.27
			10.895		0.998	0.938	1.0208	+1.801	+30.96
	7	6	19.666	633.6	1.024	0.884	1.0179	+1.194	+34.15
			19.666		1.022	0.933	1.0194	+0.9109	+27.48
	8	6	19.666	542.9	0.985	0.894	1.0172	+0.5511	+17.88
			19.666		1.006	0.898	1.0175	+0.5263	+17.52
			10.895		0.991	0.919	1.0175	+0.939	+17.46
			10.895		0.984	0.950	1.0175	+0.866	+16.53

3. Impurities

This correction was designated δk_N since nitrogen was the only impurity found in the system which requires any correction. This small correction accounts for neutrons absorbed in nitrogen in these systems. The method is discussed in the Appendices.

4. Neutron Energy Spectrum

This correction was designated δk_F and accounts for the effect on k_∞, of the change in the neutron energy spectrum caused by absorptions in the aluminum of the containment vessels.

We ask "What would the measured Δk be if the aluminum had no absorption cross-section, or if just the right amount of aluminum were present to make the ratio of the fast flux to slow flux just the correct value for the case where Δk_∞ is greater than zero ?" Let ϕ_1 be the fast flux, ϕ_2 the slow flux, m_1 the fast adjoint flux, and m_2 the slow adjoint flux.(3) Then to solve this problem, we must know how both $\frac{\phi_1}{\phi_2}$ and $\frac{m_1}{m_2}$ change as the absorber at the edge of the mixture changes. From experiments with a six-inch buffer region, it was evident that the flux ratio in the test cell did not change appreciably as the external loading was changed. It is not unreasonable to assume that the adjoint ratios behave similarly. Both of these ratios are close to the ratios of the system having the mixture plus the aluminum present. Then adding extra thermal poison to the walls of the cell should increase both $\frac{\phi_1}{\phi_2}$ and $\frac{m_1}{m_2}$. The first approximation for the changes in these ratios should then increase $\frac{\phi_1}{\phi_2}$ and $\frac{m_1}{m_2}$. Thus the approximation

$$\frac{\phi_1}{\phi_2} = \left(\frac{\phi_1}{\phi_2}\right)_o \left[1 + \alpha A\right]$$

$$\frac{m_1}{m_2} = \left(\frac{m_1}{m_2}\right)_o \left[1 + \gamma A\right]$$

where $\left(\frac{\phi_1}{\phi_2}\right)_o$ and $\left(\frac{m_1}{m_2}\right)_o$ refer to the case of no absorptions in the aluminum, α and γ are constants, and A is the number of absorptions.

A first approximation for the changes in these ratios can be derived from the two-group formalism where the thermal group neutron balance equation is written

$$D_2 \nabla^2 \phi_2 - \Sigma_{a2} \phi_2 + p \, \Sigma_1 \phi_1 = 0$$

where

D_2 is the thermal group diffusion coefficient

Σ_{a2} is the thermal group macroscopic absorption cross section

Σ_1 is the fast group macroscopic removal cross section

p is the resonance escape probability.

Assume that $\nabla \phi_2 = 0$ somewhere in the wall of the core tank and integrate over the volume enclosed by a surface passing through that position. Then

$$D_2 \int_{\text{surface}} \nabla \phi_2 \cdot dA - \Sigma_{a2c} \bar{\Phi}_{2c} V_c - \Sigma_{2p} \bar{\phi}_{2p} V_p + p \Sigma_{1c} \bar{\phi}_{1c} V_c = 0$$

Where the subscript C refers to the core material, the subscript p to the containment wall (p for poison) and V to the volume. Then

$$\frac{\phi_{1c}}{\phi_{2c}} = \frac{1}{\frac{p \Sigma_{1c}}{\Sigma_{a2c}} \left(1 - \frac{\Sigma_{a2p} \bar{\Phi}_{2p} V_p}{p \Sigma_{1c} \phi_{1c} V_c} \right)}$$

Then for small α A_p, we may use

$$\frac{\phi_{1c}}{\phi_{2c}} \simeq \frac{1}{\left(\frac{\phi_{2c}}{\phi_{c}}\right)_o \left(1 + \alpha A_p\right)}$$

Similarly,

$$\frac{m_{1c}}{m_{2c}} = \frac{1}{\left(\frac{m_1}{m_2}\right)_o \left(1 + \gamma A_p\right)} ,$$

and since the fast flux, ϕ_{1c}, is not appreciably effected by adding small amounts of poison,

then

$$\alpha = \frac{1}{p \Sigma_{1c} \bar{\phi}_{1c} V_c}$$

which is a constant.

Let A_p be the absorptions in the core tank wall (poison),

and let A_1 = absorptions in the core tank wall with no copper present

A_2 = absorptions in aluminum and some copper

A_∞ = absorptions in aluminum and copper when the flux ratio in the core is that of the correctly poisoned system.

The expression for the error in Δk_∞ is derived in the Appendix from the two group formalism. This expression is,

$$(\Delta k_\infty)_{\text{Error}} = -k_\infty \frac{\left[\left(\frac{\phi_1}{\phi_2}\right)_1 - \left(\frac{\phi_1}{\phi_2}\right)_\infty\right]\left[\left(\frac{m_1}{m_2}\right)_1 - \left(\frac{m_1}{m_2}\right)_\infty\right]}{\left(\frac{\phi_1}{\phi_2}\right)_\infty \left(\frac{m_1}{m_2}\right)_\infty}$$

Then substituting for $\frac{\phi_1}{\phi_2}$ and $\frac{m_1}{m_2}$ we have

$$\Delta k_{\infty \text{ Error}} = -k_{\infty} \frac{\gamma \alpha \left(A_1 - A_{\infty}\right)}{\left(1 - \alpha A_1\right)\left(1 - \gamma A_1\right)}$$

The curve of $(\Delta k_{\infty})_{\text{Error}}$ versus A has a maximum at $A = A_{\infty}$, and if the various constants were known, we could obtain A_{∞} and thus Δk_{∞}.

In this case there is not sufficient data with great enough precision to determine the curve's parameters. At best, a linear extrapolation to A_{∞} can be made from the points determined by A_1 and A_2; this will give an over correction - at least an upper limit on the correction.

Thus:

$$\frac{A_1 - A_{\infty}}{A_2 - A_1} = \frac{\Delta k'_{\infty_1} - \Delta k_{\infty}}{\Delta k'_{\infty_2} - \Delta k'_{\infty_1}} \simeq \frac{\delta k_F}{\Delta k'_{\infty_1} - \Delta k'_{\infty_2}}$$

The calculations of the absorption in the metal are shown in the Appendices.

Table II is a summary showing the corrections δk_T, δk_N and δk_F and the corrected values of Δk_{∞} and k_{∞}.

The general shape of the k_{∞} versus H/U ratio curves can be predicted from theory. Thus, we may use least squares technique to fit the experimental data to the curves and hence establish confidence limits on the curves. The method of making the least squares fit is to divide the experimental value of k_{∞} by the theoretical calculation of ηf, thus obtaining an "experimental" value of ϵp. These values are then fitted to the theoretical curve of ϵp. This particular method was chosen because the calculated values of η and f are quite accurate; whereas, calculated values of ϵ and p are somewhat questionable, however, the theoretical form of ϵp can be predicted.

TABLE II

SUMMARY OF $\Delta k_\infty'$

Material		Buffer Thickness			CORRECTIONS δk_N, δk_T, δk_F AND FINAL RESULTS							
% U^{235}	H/U	In Inches	$\Delta k'_\infty \times 10^3$	$\sigma(\Delta k'_\infty) \times 10^3$	$\delta k_N \times 10^3$	$\delta k_T \times 10^3$	$\delta k_F \times 10^3$	$\Delta k_\infty \times 10^{-3}$	k_∞	$\sigma(\Delta k_\infty) \times 10^3$	$\sigma_{90}(U^{235})$	$\sigma_{90}(H/U)$
1.0059	3.772	6	- 14.1	+ 3.0 - 7.1	+ 0.8	+ 1.8	+ 3.5	- 8.0	0.9920	+ 4.2 - 7.8	0.0013	0.059
	4.999	6	- 12.9	± 4.0	+ 0.4	+ 1.6	+ 3.4	- 7.5	0.9925	± 5.0		0.066
	6.164	6	- 13.3	± 5.0	+ 0.4	- 3.0	+ 3.4	- 12.5	0.9875	± 5.8		0.055
	6.881	6	- 23.1	+ 4.7 - 4.2	+ 0.9	- 0.2	+ 4.5	- 17.9	0.9821	+ 5.6 - 5.2		0.057
	7.449	2	- 14.3	+ 5.8 - 12.9	+ 0.9	+ 2.1	+ 2.2	- 9.1	0.9909	+ 7.6 - 13.8		0.058
	7.449	4	- 33.3	± 7.0	+ 0.9	+ 2.1	+ 3.0	- 27.3	0.9727	± 7.6		0.058
	7.449	6	- 39.5	± 5.3	+ 0.9	+ 2.1	+ 6.7	- 29.8	0.9702	± 7.0		0.058
1.0704	3.728	6	+ 3.7	+ 5.9 - 8.2	+ 1.0	- 0.2	+ 1.8	+ 6.3	1.0063	+ 6.2 - 8.4	0.0012	0.050
	5.778	6	+ 3.6	+ 6.1 - 9.0	+ 0.9	+ 0.1	+ 1.8	+ 6.4	1.0064	+ 6.4 - 9.2		0.054
	7.075	6	- 7.8	± 5.8	+ 0.7	-	+ 2.8	- 4.3	0.9957	± 6.1		0.203
1.1586	3.728	6	+ 30.5	+ 3.2 - 7.7	+ 0.9	- 0.8	- 0.8	+ 29.8	1.0298	+ 3.4 - 7.8	0.0016	0.050
	5.926	6	+ 31.2	± 5.0	+ 0.8	+ 1.8	- 0.8	+ 33.0	1.0330	± 5.1		0.074
	6.838	6	+ 32.5	+ 3.4 - 2.7	+ 0.6	- 0.8	- 1.0	+ 31.3	1.0313	+ 3.5 - 2.9		0.104
	7.449	6	+ 17.0	± 5.0	+ 0.9	+ 2.0	+ 1.0	+ 20.9	1.0209	± 5.1		0.058

In addition, since the range of U^{235} enrichment was very small for these experiments, the values of ϵ and p are independent of enrichment and hence all "experimental" values of ϵ p should fall on a smooth curve predicted by the theoretical calculation of ϵ p. The variation of the "experimental" values of ϵ p from this curve is then a worthwhile check on the reliability of the experimental data.

The following method was used:

Let γ = H/U atomic ratio.

The quantity, p, is then given by, (4)

$$p = \exp. - a\, g(y)$$

$$\text{where } g(y) = \frac{1}{\bar{\xi}} \left(\frac{N_{238}}{\Sigma_s} \right)^{0.585}$$

and "a" is a constant

$$\bar{\xi} = \frac{\sum_i \left(\xi N \sigma_s \right)_i}{\sum_i \left| N\, \sigma_s \right|_i}$$

$$\Sigma_s = \sum_i \left| N\, \sigma_s \right|_i$$

N = number of atoms/cm^3

σ_s = microscopic scattering cross-section

ξ = average logrithmatic energy decrement per collision.

The form of ϵ (taken from the calculations of ϵ shown in the theoretical calculations), is

$$\epsilon = 1 + \frac{b}{y + 0.5} \simeq \exp. \frac{b}{y + 0.5} \quad \text{for } \frac{b}{y + 0.5} << 1$$

Then we have

$$\varepsilon p = \exp. -a\, g(y) + \frac{b}{y + 0.5}$$

Taking logrithims and multiplying by (y + 0.5) gives,

$$(y + 0.5) \ln \varepsilon p = -a\, g(y)\, (y + 0.5) + b.$$

Let $Z = (y + 0.5) \ln \varepsilon p$

$X = g(y)\, (y + 0.5)$

The least squares fit is then made to the straight line,

$$Z = -aX + b$$

The experimental points to be fitted are found by

$$\varepsilon p = \frac{(k_\infty) \text{ experimental}}{(\eta f) \text{ calculated}}$$

Figures 7, 8 and 9 show the results of this technique.

Minimum U^{235} Enrichment for Criticality

The minimum U^{235} enrichment required for criticality in UO_3-water mixtures can be found from the k_∞ versus H/U curves. It is the enrichment for which the maximum value of k_∞, on the k_∞ versus H/U ratio curve, is just unity.

The value was determined by the interpolation of maximum values of k_∞ as functions of enrichment, where these values were obtained from the k_∞ versus H/U ratio curves of Figure 9. Figure 10 shows the curve of the maximum value of k_∞ versus enrichment.

The minimum U^{235} enrichment for criticality in UO_3-water systems was found to be $1.034\,^{+0.010}_{-0.009}$ where the errors are representative of 90 per cent confidence limits.

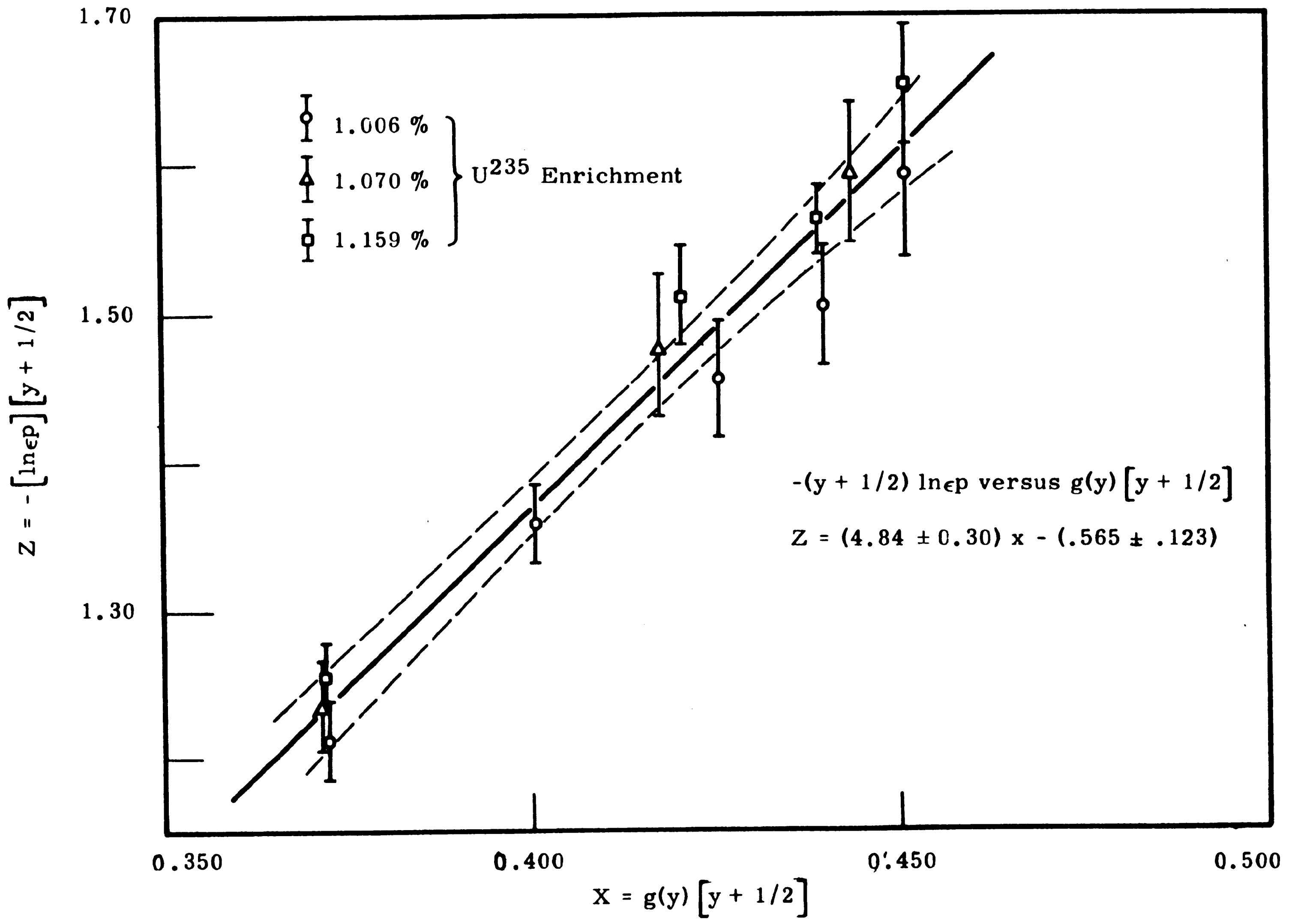


FIGURE 7

Plot of Z Versus X for Comparison of Experimental and Theoretical Values of ϵp

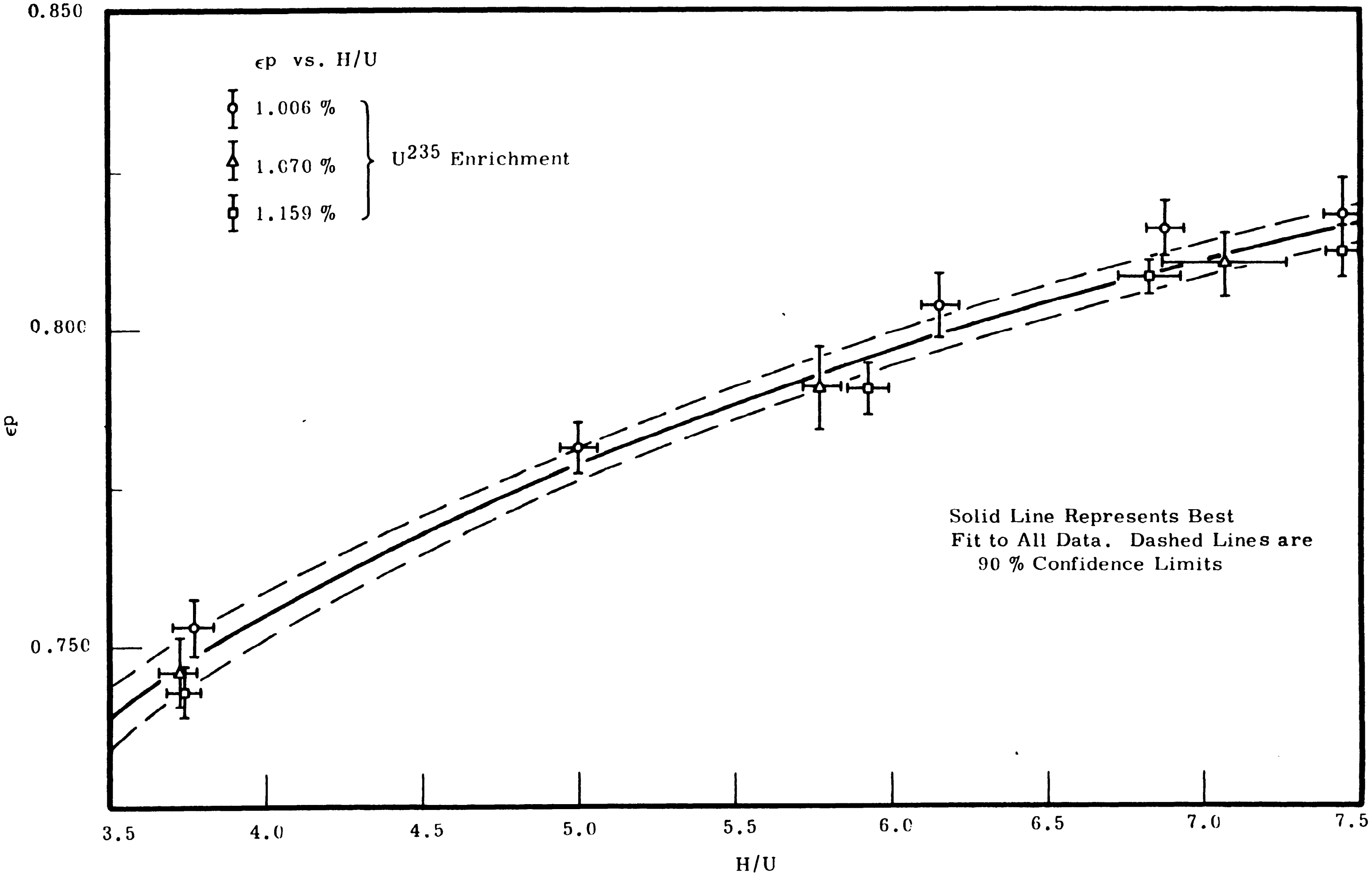
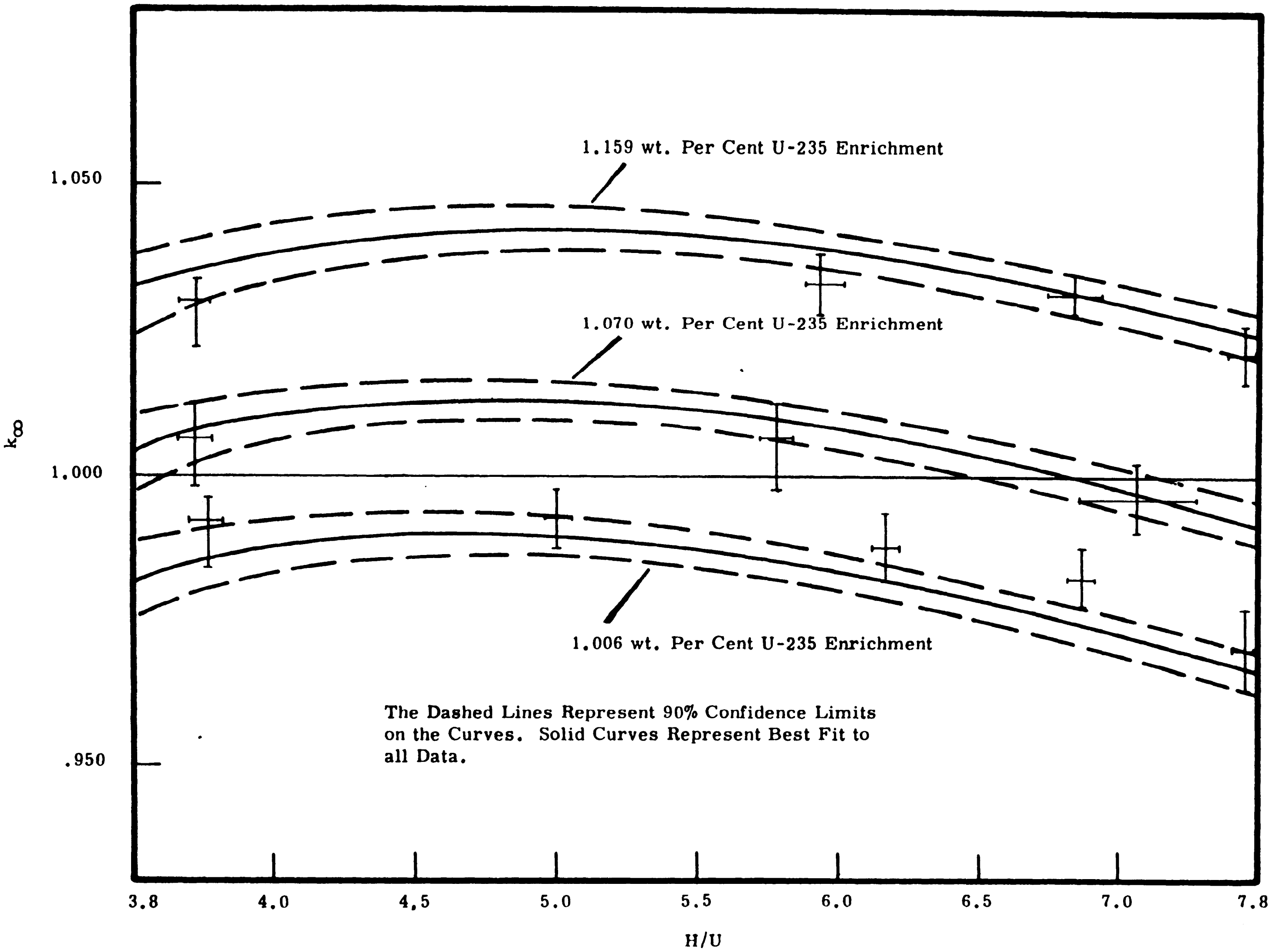


FIGURE 8

Plot of εp Versus H/U for Comparison of Experimental and Theoretical Values of εp



The Dashed Lines Represent 90% Confidence Limits on the Curves. Solid Curves Represent Best Fit to all Data.

FIGURE 9

k_∞ Versus H/U Atomic Ratio for Slightly Enriched Homogeneous UO_3 - Water Mixtures

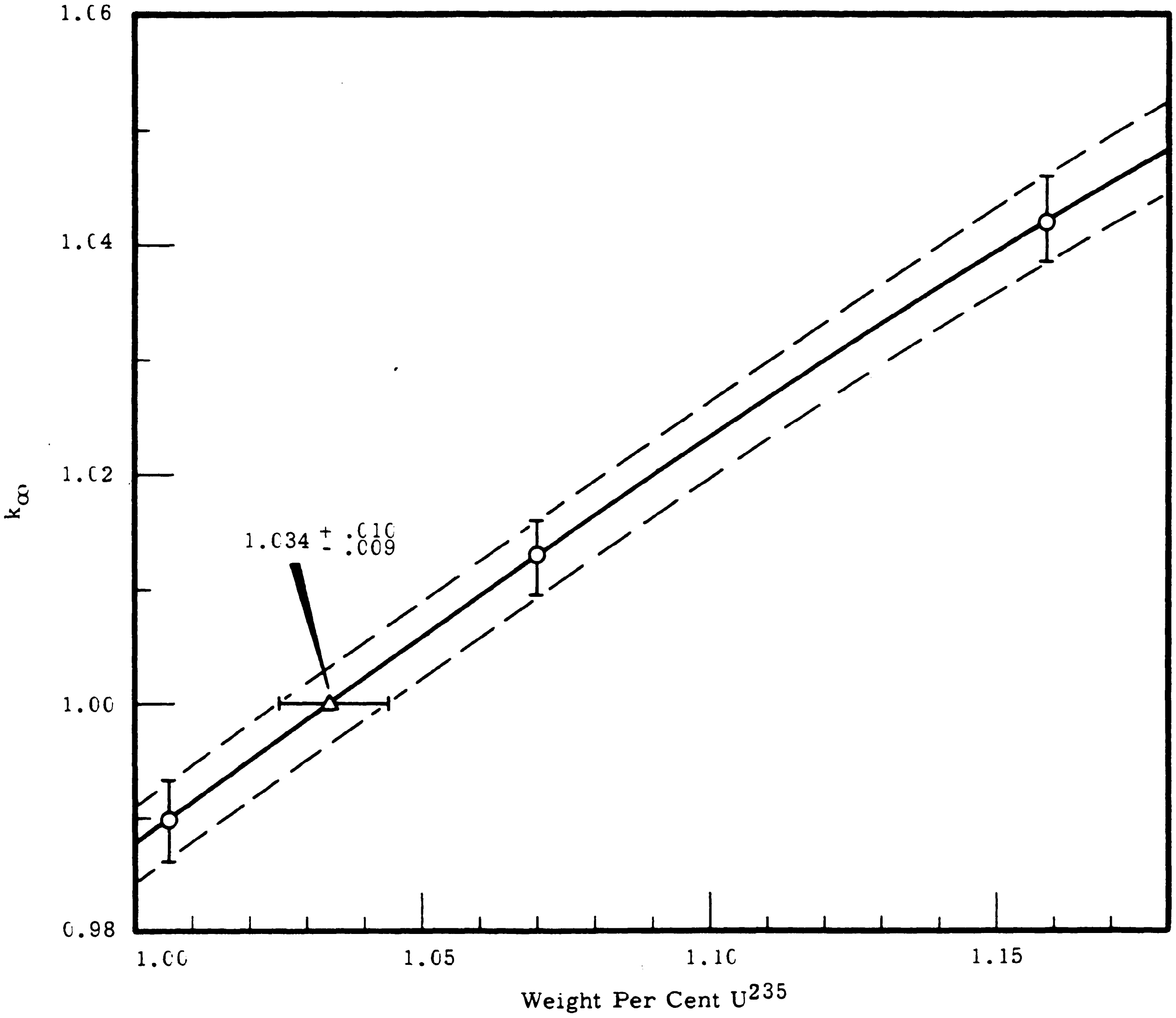


FIGURE 10

Minimum U^{235} Enrichment for Criticality in UO_3 - Water Mixtures

Theoretical Calculations of k_∞ (Four Factor Formula)

These experiments on low enrichment UO_3-water mixtures were originally undertaken because of the lack of agreement between theoretical estimates of the minimum enrichment for criticality in UO_3-water mixtures.

The calculations of k_∞ presented here are somewhat improved over the original estimates for these simple four-factor formula calculations.

In the four-factor formula

$$k_\infty = \eta \varepsilon p f$$

The terms ε and p give the greatest difficulty, and a number of forms are presented here.

1. Calculations of η (Glasstone and Edlund formula) (4)

$$\eta = \frac{\left(\nu \bar{\Sigma}_f\right)_{235}}{\left(\bar{\Sigma}_a\right)_{235} + \left(\bar{\Sigma}_a\right)_{238}}$$

$$\bar{\Sigma} = \frac{N_o \bar{\sigma} f_{1/v} M}{A V}$$

$$\bar{\sigma} = \frac{\sqrt{\pi}\, \sigma_{2200}}{2}$$

M = mass
V = volume
$f_{1/v}$ = thermal "non-1/v" factor
A = atomic or molecular weight
N_o = Avogadro's number

2. Calculation of f (Glasstone and Edlund formula)(4)

$$f = \frac{(\bar{\Sigma}_a)_{235} + (\bar{\Sigma}_a)_{238}}{(\bar{\Sigma}_a)_{235} + (\bar{\Sigma}_a)_{238} + (\bar{\Sigma}_a)_{O_3} + (\bar{\Sigma}_a)_{H_2O}}$$

3. Calculation of p (Glasstone and Edlund formula) (4)

$$p = \exp. - \frac{3.9}{\bar{\xi}} \left(\frac{N_{238}}{\Sigma_s} \right)^{0.585}$$

where $$N_{238} = \frac{N_o}{V} \left(\frac{M}{A} \right)_{238}$$

$$\Sigma_s = \sum_i \left(\frac{N_o \sigma_s M}{V A} \right)_i$$

$$\bar{\xi} = \frac{\sum_i \left(\frac{\xi N_o \sigma_s M}{V A} \right)_i}{\sum_i \left(\frac{N_o \sigma_s M}{V A} \right)_i}$$

Figure 11 is a curve of p versus H/U ratio showing the Glasstone and Edlund results and the results due to Safanov. (4,5)

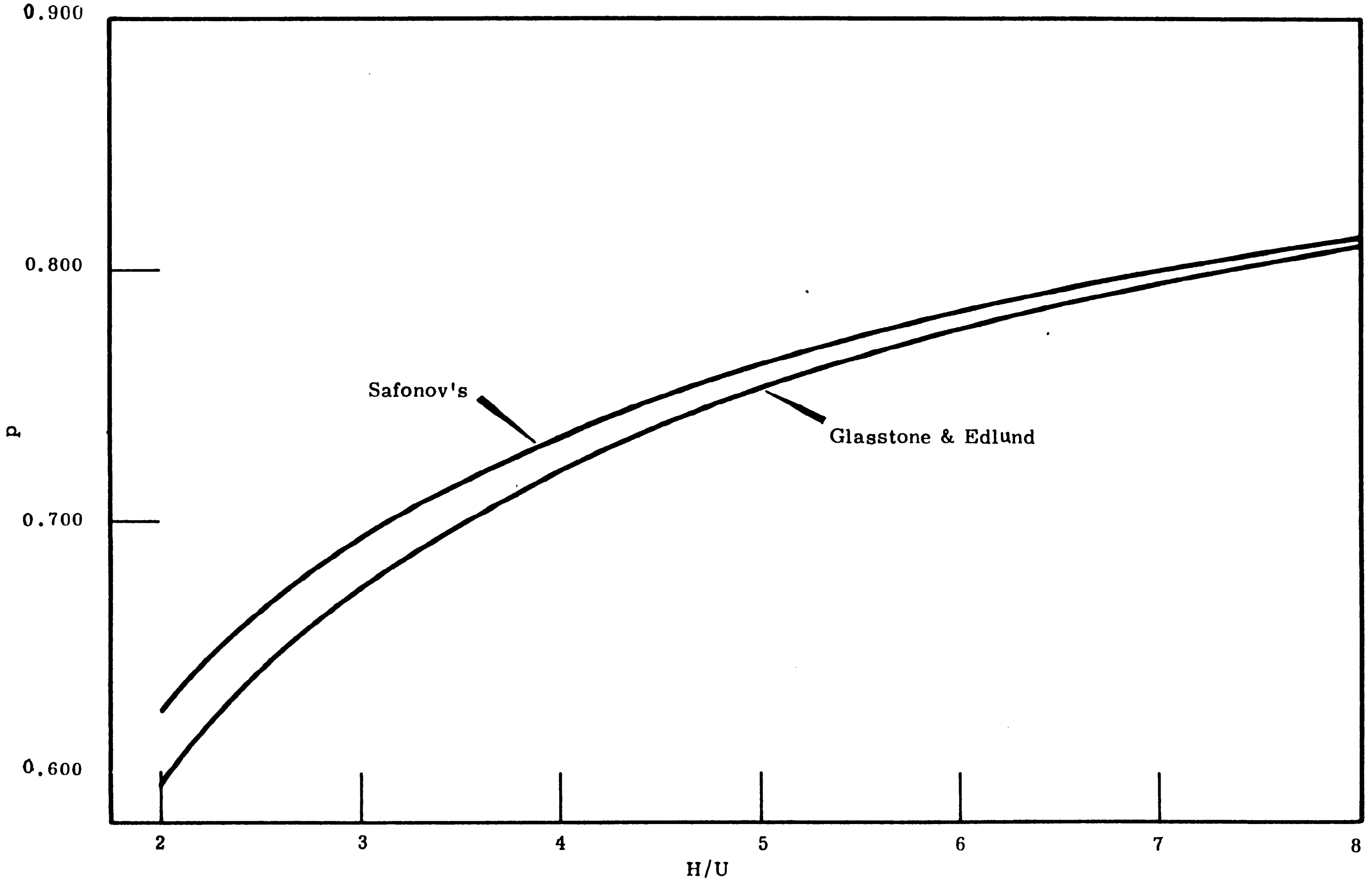


FIGURE 11

p for UO_3 - H_2O Mixtures

4. Calculation of ϵ (H. Rodrick formula) (6)

Assume:

1. that any collision with H and inelastic scattering with U reduces the energy of a neutron to below fast fission threshold.
2. that elastic collisions with U and O do not change the neutron energy.
3. that there is no inelastic scattering or absorption in O and no absorption in H.
4. that U^{235} behaves like U^{238}.

Then $\epsilon = \dfrac{\text{number of fast neutrons below threshold/primary collision}}{1 - \text{number of fast neutrons above threshold/primary collision}}$

$$\epsilon = \frac{\dfrac{N_H \sigma_{eH} + N_U \sigma_{i\,U}}{\Sigma_t}}{1 - \dfrac{N_o \sigma_{eo} + N_U \left(\sigma_e + \nu\sigma_f\right)_U}{\Sigma_t}} = 1 + \frac{\alpha}{\dfrac{N_H \sigma_H}{N_U} + (\sigma_{i\,U} - \alpha)}$$

where

Σ_t = total macroscopic cross section

$\alpha = \sigma_{fU}(\nu - 1) - \sigma_{c\,U}$

σ_c = microscopic capture cross section

σ_e = elastic microscopic cross section

σ_f = microscopic fission cross section

σ_i = inelastic microscopic cross section.

5. Calculation of ϵ: (Hellen's formula)$^{(7)}$

$$\epsilon = 1 + \frac{\alpha}{\frac{N_O}{N_{238}}\left|\sigma_c + \sigma_i\right|_O + \frac{N_H}{N_{238}}\left|\sigma_c + \sigma_i\right|_H + \left|\sigma_i - \alpha\right|_{238}}$$

Figures 12, 13, and 14 show the results of these various calculations.

ERROR ANALYSIS

Propagation of error techniques were used wherever applicable in determining the error in the experimental values of $\Delta k'_\infty$. The sources of experimental error were from reactivity measurements, counting of activated foils, mass spectrometer measurements of U^{235} enrichment, chemical analysis of water content, and weights of the various materials. In addition there were the standard errors in the cross section, non-1/v factors and atomic weights.

Tables III through VI are summaries of some of the various errors. The formulas for calculating these values are contained in the Appendices. The following definitions are used in Tables III through VI.

γ = fractional U^{235} enrichment $\left(\frac{M_{U-235}}{M_{U-235} + M_{U-238}}\right)$

δ = fractional weight of water in the UO_3 - water mixture $\left(\frac{M_{H_2O}}{M_{H_2O} + M_{UO_3}}\right)$

d = density

M_c = mass of the core tank materials

V_c = volume of the core tank

n = number of independent measurements.

The function f^2 (X) is defined as

$$f^2(X) = \frac{\sigma(X)^2}{X^2}$$

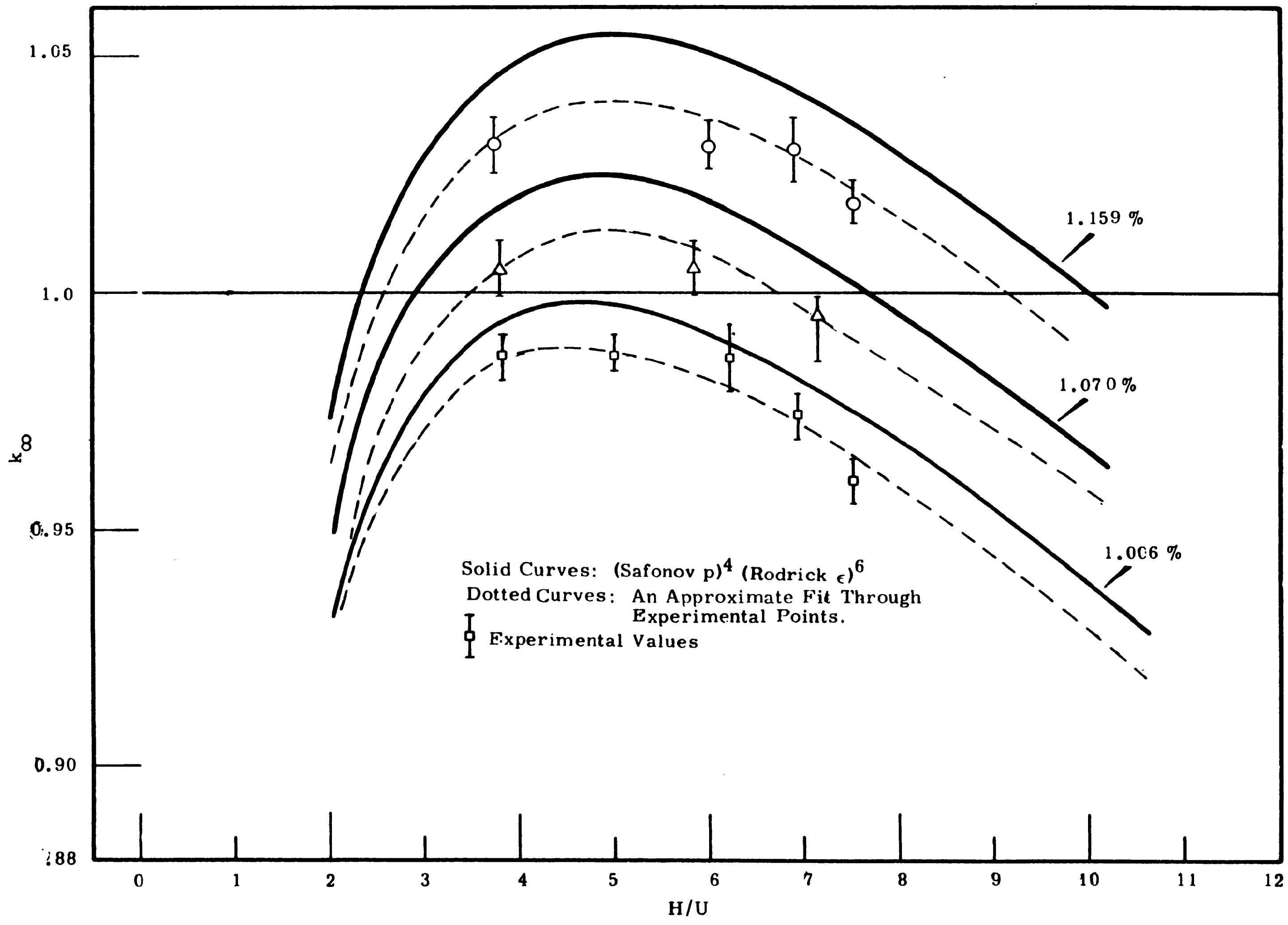


FIGURE 12

k∞ of Enriched UO_3 - H_2O Mixtures

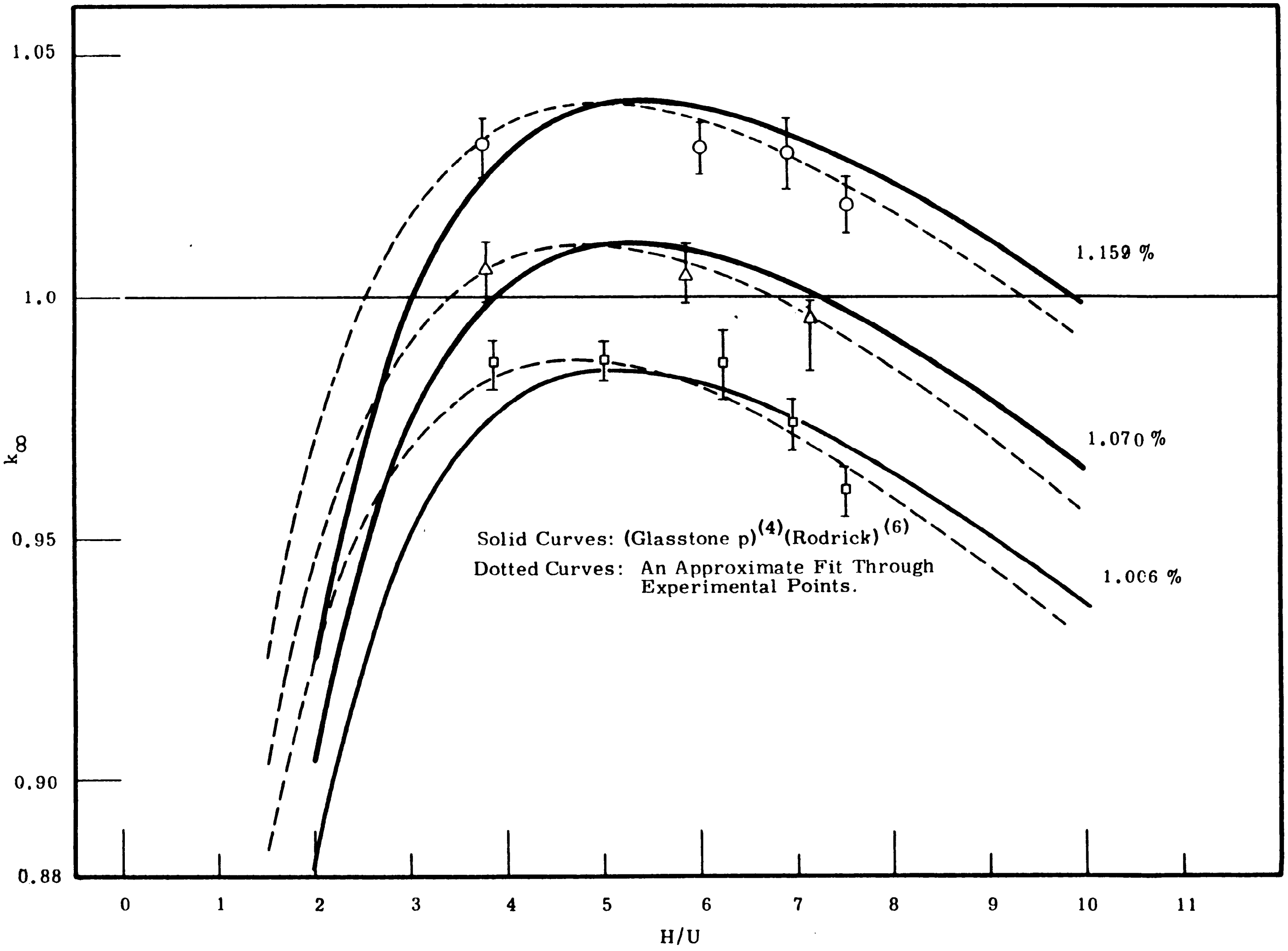


FIGURE 13

k ∞ of Enriched UO_3 - H_2O Mixtures

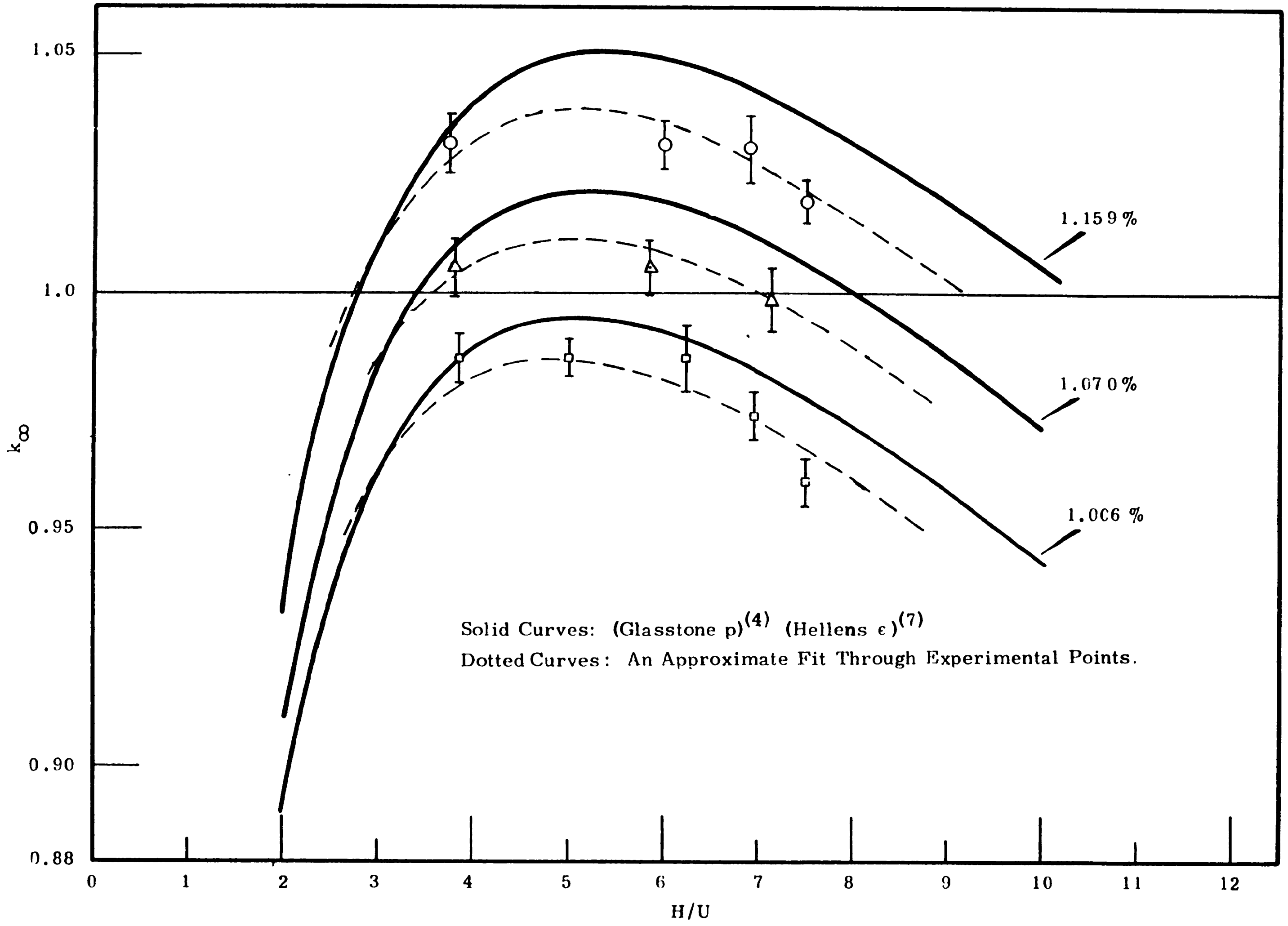


FIGURE 14

k_∞ of Enriched UO_3 - H_2O Mixtures

TABLE III

CALCULATION OF

$$\sigma^2 \frac{\Delta\rho_{eV}}{\Delta\rho_{cp}}, \quad \sigma^2 \frac{\bar{\phi}_p \bar{\phi}_{pp}}{\bar{\phi}_c \bar{\phi}_V} \text{ and } \sigma^2 \Delta k'_\infty$$

Material									
% U^{235}	H/U	Buffer Thickness In Inches	$\left(\frac{\Delta\rho_{eV}}{\Delta\rho_{cp}}\right)^2$ $\times 10^2$	$\sigma^2\left(\frac{\Delta\rho_{cV}}{\Delta\rho_{cp}}\right)$ $\times 10^4$	$f^2\left(\frac{\Delta\rho_{cV}}{\Delta\rho_{cp}}\right)$ $\times 10^4$	$f^2\left(\frac{\phi_p \phi_{pp}}{\phi_c \phi_V}\right)$ $\times 10^4$	$f^2(\Delta k'_\infty)$ $\times 10^4$	$\sigma^2(\Delta k'_\infty)$ $\times 10^6$	$\sigma(\Delta k'_\infty)$ $\times 10^3$
1.006	4	6	11.97	1.904	15.91	2.31	18.22	0.1873	0.43
			20.52	3.063	14.93		17.24	0.3634	0.60
	5	6	23.04	1.434	6.223		8.53	0.1356	0.37
			22.94	1.786	7.785		10.10	0.1555	0.39
			26.63	1.417	5.321		7.63	0.1404	0.38
			32.72	0.8613	2.632		4.94	0.1176	0.34
	6	6	22.28	2.680	12.03		14.34	0.2256	0.48
			27.56	2.332	8.461		10.77	0.2099	0.46
			99.00	16.94	17.11		19.42	0.4581	0.68
			119.90	16.01	13.35		15.66	0.4557	0.68
	7	6	36.97	2.455	6.640		8.95	0.3925	0.63
			84.09	3.725	3.959		6.27	0.7091	0.84
	8	2	0.08703	0.6705	770.4		772.7	0.07974	0.28
			16.67	0.7845	4.706		7.02	0.1438	0.38
	8	4	62.09	5.765	9.284		11.59	0.8090	0.90
			94.87	1.240	1.307		3.62	0.4007	0.63
			166.41	9.393	5.644		7.95	0.5222	0.72
			324.36	9.008	2.777		5.09	0.6476	0.81
	8	6	105.47	4.400	4.171		6.48	0.8259	0.91
			169.00	4.848	2.868		5.18	0.9672	0.98
			393.63	39.41	10.01		12.32	1.904	1.37
			438.48	31.63	7.213		9.52	1.621	1.27
1.070	4	6	9.734	1.462	15.02	2.31	17.33	0.1911	0.44
			0.0484	1.590	3285.		3287.	0.1690	0.41
	6	6	0.6989	8.840	126.5		128.8	0.0925	0.30
			13.99	1.142	8.162		10.47	0.1605	0.40
	8	6	1.416	1.350	95.34		97.65	0.1309	0.36
			14.59	1.937	13.28		15.59	0.2200	0.47
1.159	4	6	289.0	18.43	6.377		8.69	0.7634	0.87
			108.2	1.326	1.225		3.54	0.3940	0.63
	6	6	109.6	1.014	0.9251		3.24	0.3144	0.56
			107.5	1.234	1.148		3.46	0.3381	0.58
			358.0	9.533	2.663		4.97	0.5501	0.74
			324.4	10.19	3.141		5.45	0.5224	0.72
	7	6	142.6	1.177	0.8253		3.14	0.3662	0.61
			82.97	0.8400	1.0124		3.32	0.2507	0.50
	8	6	30.37	0.9310	3.066		5.38	0.1720	0.42
			27.70	0.7063	2.550		4.86	0.1492	0.39
			88.17	4.058	4.602		6.91	0.2107	0.46
			75.00	3.268	4.357		6.67	0.1822	0.43

TABLE IV

CALCULATION OF $\sigma(\bar{\gamma})$

$\bar{\gamma} = 0.01006$

Nominal H/U	δ_i	$\bar{\delta}$	$(\delta_i - \bar{\delta}) \times 10^4$	$\sigma^2(\delta_i) \times 10^8$	$\sigma^2(\bar{\delta}) \times 10^8$	$\sigma(\delta_i)$	$\sigma(\bar{\delta})$
4	0.1080	0.1077	3	146	29	0.0012	0.00054
	0.1067		10				
	0.1062		15				
	0.1092		15				
	0.1082		5				
5	0.1374	0.1375	1	66	22	0.0008	0.00047
	0.1368		7				
	0.1384		9				
6	0.1641	0.1640	1	2	0.5	0.0001	0.00007
	0.1642		2				
	0.1640		0				
	0.1638		2				
	0.1640		0				
7	0.1803	0.1795	8	44	8.8	0.0007	0.00030
	0.1790		5				
	0.1802		7				
	0.1794		1				
	0.1789		6				
8	0.1905	0.1914	9	55	11.0	0.0007	0.00033
	0.1915		1				
	0.1925		11				
	0.1910		4				
	0.1915		1				

TABLE IV (Contd.)

$\overline{\gamma} = 0.01070$

Nominal H/U	δ_i	$\overline{\delta}$	$(\delta_i - \overline{\delta})$ $\times 10^4$	$\sigma^2(\delta_i)$ $\times 10^8$	$\sigma^2(\overline{\delta})$ $\times 10^8$	$\sigma(\delta_i)$	$\sigma(\overline{\delta})$
4	0.1055	0.1064	9	50	10	0.0007	0.00031
	0.1061		3				
	0.1065		1				
	0.1074		10				
	0.1067		3				
6	0.1558	0.1554	4	19	3.8	0.0004	0.00020
	0.1556		2				
	0.1548		6				
	0.1556		2				
	0.1550		4				
8	0.1870	0.1836	34	1581	316	0.0040	0.0018
	0.1802		34				
	0.1884		48				
	0.1795		41				
	0.1831		5				
			$\overline{\nu} = 0.01159$				
4	0.1065	0.1066	1	5.5	1.8	0.00024	0.00014
	0.1065		1				
	0.1069		3				
6	0.1567	0.1587	20	207	41	0.0014	0.00064
	0.1590		3				
	0.1590		3				
	0.1606		19				
	0.1580		7				
7	0.1776	0.1786	10	404	81	0.0020	0.00090
	0.1821		35				
	0.1774		12				
	0.1787		1				
	0.1774		12				
8	0.1909	0.1914	5	42	8.3	0.0006	0.00029
	0.1915		1				
	0.1912		1				
	0.1908		6				
	0.1924		10				

TABLE V

CALCULATION OF $\sigma(\bar{v})$

Per Cent Nominal Enrichment	$\bar{v}$	$\sigma(v)$	n	$\sigma(\bar{v})$
1.00	0.010059	0.000039	27	0.0000075
1 07	0.010704	0.000016	7	0.0000060
1.15	0.011586	0.000036	15	0.0000093

TABLE VI

CALCULATION OF $\sigma(M_c)$ AND $\sigma(d)$

$\bar{v}$	Nominal H/U	$f^2(M_c)$ $\times 10^8$	$f^2(V_c)$ $\times 10^8$	$f^2(d)$ $\times 10^8$	$\sigma^2(d)$ $\times 10^8$	$\sigma(M_c)$ (gm)	$\sigma(d)$ (gm)
0.01006	4	9.21	507.2	516.4	2061	5.4	0.0045
	5	6.99		514.2	2704		0.0052
	6	7.38		514.6	2564		0.0051
	7	12.43		519.6	1537		0.0039
	8	12.34		519.5	1548		0.0039
0.01070	4	11.64	507.2	518.8	1640		0.0041
	6	11.48		518.7	1662		0.0041
	8	10.84		518.0	1758		0.0042
0.01159	4	11.40	507.2	518.6	1673		0.0041
	6	11.01		518.2	1734		0.0042
	7	10.05		517.3	1893		0.0044
	8	13.67		520.9	1401		0.0037

APPENDIX I

THEORETICAL DESCRIPTION OF THE PCTR EXPERIMENT

In the four-factor formula $k_\infty = \eta f \epsilon p$, absorptions in the thermal and epithermal regions enter into the calculation of these factors. If the flux is separated into two groups by a definite cut-off energy, η must depend on both groups in order to give the correct number of fission neutrons (due to the contribution of epithermal fissions), and the addition of a "$\frac{1}{v}$" absorber will affect both f and p. (Since p must include all absorptions above cutoff.) In order to remove the dependence of p on the addition of a "$\frac{1}{v}$" absorber, the concept of a "$\frac{1}{v}$" parameter is introduced. The thermal utilization (f) and η include not only the thermal absorptions but also the "$\frac{1}{v}$" part of the epithermal absorptions. The resonance escape probability (p) is then truly a "resonance escape" parameter. The only omission is that portion of the fission neutrons coming from the resonance fission of U^{235} ("$\frac{1}{v}$" fissions are included in η).

The following definitions hold:

ϵ = Number of fast neutrons produced by all fissions per fast neutron produced by U^{235} fissions.

$1 - p$ = Number of fast neutrons absorbed while slowing down per fast neutron produced by all fissions.

or p = Number of neutrons reaching thermal per fast neutron produced by all fissions.

γ = Number of fast neutrons produced by epithermal fission of U^{235} per fast neutron absorbed while slowing down.

η = Number of fast neutrons produced by thermal fission of U^{235} per thermal neutron absorbed in uranium.

f = Number of thermal neutrons absorbed in uranium per thermal neutron absorbed in all materials.

In terms of cross sections: (for a homogeneous system)

$$f = \frac{\Sigma_{U2}}{\Sigma_2}$$

$$\eta = \frac{\nu \Sigma_{f2}}{\Sigma_{U2}}$$

$$\gamma = \frac{\nu \Sigma_{f1}^{235}}{\Sigma_{a1}}$$

$$p = \frac{\Sigma_{sd}}{\Sigma_1} \qquad \Sigma_1 = \Sigma_{sd} + \Sigma_{a1}; \quad \Sigma_{a1} = \Sigma_{c1} + \Sigma_{f1}^{235}$$

where

Σ_1 = Removal cross section for fast neturons.

Σ_{sd} = "Slowing down" cross section for fast neutrons

Σ_{1R} = Absorption cross section for fast neutrons.

Σ_2 = Absorption cross section for slow neutrons.

k_∞ = Infinite multiplications constant = $\eta \epsilon p f$

$\Sigma_1 = \Sigma_{1R} + \Sigma_{sd} = \frac{\Sigma_{sd}}{p}$

Let us consider an infinite, homogeneous multiplying system characterized by the following constants and variables:

$$\phi_{1\infty} = \text{"fast flux"} = \int_{E_{Rc}}^{E_{max}} \phi(E)\, dE = \int_{E_{Rc}}^{E_{max}} \frac{\phi_{r\infty}}{E}\, dE = \phi_{r_\infty} \ln \frac{E_{max}}{E_{Rc}}$$

for a "$\frac{1}{E}$" fast flux /unit energy and where ϕ_{r_∞} = constant.

$$\phi_{2\infty} = \text{"Slow flux"} = \int_0^\infty \phi_{th}(E)\,dE = \phi_{th\,\infty} \int_0^\infty \frac{E}{kT} \exp \cdot -\frac{E}{kT} \frac{dE}{kT}$$

for a Maxwellian slow flux.

$$\phi_{th_\infty} = \bar{v}_{th_\infty} n_{th_\infty} = \left(\frac{2}{\sqrt{\pi}}\right)\left(\sqrt{\frac{2kT}{M}}\right)\left(n_{th\infty}\right)$$

Note that,

$$\Sigma_2 = \frac{\int \Sigma_2(E)\,\phi_{th}(E)\,dE}{\phi_{th\,\infty}}$$

$$\Sigma_{1R} = \frac{\int_{E_{Rc}}^{E} \Sigma(E)\,\phi(E)\,dE}{\int_{E_{Rc}}^{E_{max}} \phi(E)\,dE}$$

Let us poison the system to $k'_\infty = 1$:

The diffusion equations are:

$$-\Sigma_1 \phi_{1\infty} + \varepsilon \nu \Sigma_{zf} \phi_{2\infty} = 0$$

$$p\,\Sigma_1 \phi_{1\infty} - \Sigma'_2 \phi_{2\infty} = 0$$

where

$$\Sigma'_2 = \Sigma_2 + \Sigma_{2p}$$

and Σ_{2p} = additional slow absorption cross section needed to poison the system to $k'_\infty = 1$ (poison is homogeneous).

The adjoint equations are:

$$- \Sigma_1 m_{1\infty} + p \Sigma_1 m_{2\infty} = 0$$

$$\epsilon \nu \Sigma_{zf} m_{1\infty} - \Sigma_2' m_{2\infty} = 0$$

thus,

$$\frac{\phi_{1\infty}}{\phi_{2\infty}} = \frac{\epsilon \nu \Sigma_{2f}}{\Sigma_1} = \frac{k_\infty \Sigma_2}{p \Sigma_1}$$

and

$$\frac{m_{1\infty}}{m_2} = p$$

$$k_\infty \Sigma_2 = \Sigma_2' = \Sigma_2 + \Sigma_{2p}$$

$$k_\infty - 1 = \frac{\Sigma_{2p}}{\Sigma_2}$$

In a finite reactor, we place a cell of the poisoned material of volume v in a position such that the fluxes in the cell are flat ($\nabla \phi_{1,2} = 0$) and compare the reactivity of the reactor with this arrangement with its reactivity when the cell is replaced by a void. By two-group pertubation theory:

$$\Delta \rho = \rho_p - \rho_v = \text{(reactivity with poisoned cell) - (reactivity with void)}$$

$$= \frac{V}{\chi} \left(- \Sigma_2 m_2 \phi_2' - \Sigma_1 m_1 \phi_1' + \Sigma_{sd} m_2 \phi_1' + \epsilon \nu \Sigma_{2f} m_1 \phi_2' \right)$$

where

ϕ' = flux in material, m = adjoint flux in void

or

ϕ' = flux in void, m = adjoint flux in material and

$$\chi = \int_{\text{reactor}} (m_1 \phi'_1 + m_2 \phi'_2)\, dV$$

$$\Delta\rho = \left(\frac{V}{\chi}\right) m_2 \phi_2' \left(- \Sigma'_2 - \Sigma_1 \frac{m_1 \phi'_1}{m_2 \phi'_2} + p\, \Sigma_1 \frac{\phi'_1}{\phi'_2} + \frac{k_\infty \Sigma_2 m_1}{p\, m_2} \right) .$$

Now assume

1. $$\frac{\phi'_1}{\phi'_2} = \frac{\phi_{1\infty}}{\phi_{2\infty}} = \frac{k_\infty \Sigma_2}{p \Sigma_1}$$

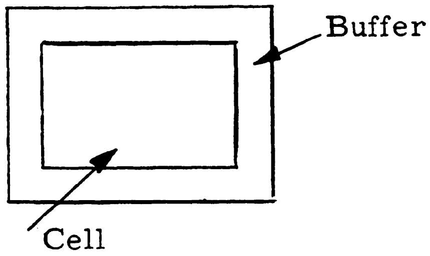
or 2. $$\frac{m_1}{m_2} = \frac{m_{1\infty}}{m_{2\infty}} = p.$$

Then

1. $$\Delta\rho = \left(\frac{V}{\chi}\right) m_2 \phi'_2 \left(- \Sigma'_2 - \frac{k_\infty \Sigma_2}{p} \frac{m_1}{m_2} + k_\infty \Sigma_2 + \frac{k_\infty \Sigma_2\, m_1}{p\, m_2} \right) = 0$$

or 2. $$\Delta\rho = \left(\frac{V}{\chi}\right) m_2 \phi'_2 \left(- \Sigma_2 - \Sigma_1\, p \frac{\phi'_1}{\phi'_2} + p\, \Sigma_1 \frac{\phi'_1}{\phi'_2} + k_\infty \Sigma_2 \right)$$

Thus, if the flux ratio $\frac{\phi'_1}{\phi'_2}$ in either the cell or the void is matched to the infinite-medium flux ratio, or if the adjoint ratio $\frac{m_1}{m_2}$ in either cell or the void is matched to an infinite-medium adjoint ratio, the reactor cannot differentiate between a cell with $k'_\infty = 1$ and a void.



The ideal experiment for measuring $k'_\infty - 1$ is:

1. Assume a correct amount of posion and add it to the cell homogeneously.
2. Surround the cell by a buffer region of similarly poisoned material.
3. Adjust the spectrum until the flux or adjoint ratios are the same in the cell as in the buffer.
4. Measure $\Delta \rho$.

Repeat 1 through 4 until $\Delta \rho = 0$. Then $k_\infty - 1$ is determined from

$$k_\infty - 1 = \frac{\Sigma_{2p}}{\Sigma_2}.$$

The value of k_∞ for heterogeneous poisoning with some resonance absorption is found as follows:

In the infinite medium the diffusion equations are

$$D_2 \nabla^2 \phi_2 - \Sigma_2 \phi_2 + p \Sigma_1 \phi_1 = 0$$

$$D_1 \nabla^2 \phi_1 - \Sigma_1 \phi_1 + \frac{k \Sigma_2}{p} \phi_2 = 0$$

and $-D_2 \nabla \phi (a) = 1/2 \; \Sigma_{2p} t \; \phi (a)$

$-D_1 \nabla \phi (a) = 1/2 \; \Sigma_{1p} \; t \; \phi (a)$

t = thickness of the poison

These equations yield, upon integration:

$$-\Sigma_{2p}\, V_p\, \bar{\phi}_{2p} - \Sigma_{2c}\bar{\phi}_{2c}\, V_c + p\, \Sigma_{1c}\bar{\phi}_{1c}\, V_c = 0$$

$$-\Sigma_{1p}\, V_p\, \bar{\phi}_{1p} - \Sigma_{1c}\,\bar{\phi}_{1c}\, V_c + \frac{k}{p}\,\Sigma_{2c}\,\bar{\phi}_{2c}\, V_c = 0$$

Thus,

$$\left(\frac{\bar{\phi}_{1c}}{\bar{\phi}_{2c}}\right)_{\infty} = \frac{\Sigma_{2c}}{p\,\Sigma_{1c}}\left(1 + \frac{\Sigma_{2p}\,\bar{\phi}_{2p}\, V_p}{\Sigma_{2c}\,\bar{\phi}_{2c}\, V_c}\right) = \frac{k_{\infty}}{p}\;\frac{\Sigma_{2c}}{\Sigma_{1c}}\left(1 - \frac{p\,\Sigma_{1p}\bar{\phi}_{1p}\, V_p}{k_{\infty}\Sigma_{2c}\,\bar{\phi}_{2c}\, V_c}\right)$$

and

$$k_{\infty} = \frac{f\,p}{f'\,p'} = 1 + \frac{\Sigma_{2p}\bar{\phi}_{2p}\, V_p}{\Sigma_{2c}\,\bar{\phi}_{2c}\, V_c} + p\,\frac{\Sigma_{1p}\,\bar{\phi}_{1p}\, V_p}{\Sigma_{2c}\,\bar{\phi}_{2c}\, V_c}$$

where

$$f = \frac{\Sigma_{2U}}{\Sigma_{2c}}\;;\quad f' = \frac{\Sigma_{2U}\,\bar{\phi}_{2c}\, V_c}{\Sigma_{2c}\bar{\phi}_{2c}\, V_c + \Sigma_{2p}\bar{\phi}_{2p}\, V_p}$$

$$p = \frac{\Sigma_{sd}}{\Sigma_{1c}}\;;\quad p' = \frac{\Sigma_{sd}\,\phi_{1c}\, V_c}{\Sigma_{1c}\,\bar{\phi}_{1c}\, V_c + \Sigma_{1p}\,\phi_{1p}\, V_p}$$

$$= \frac{-\;p\,\Sigma_{1c}\,\bar{\phi}_{1c}\, V_c}{\Sigma_{1c}\,\bar{\phi}_{1c}\, V_c + \Sigma_{1p}\,\bar{\phi}_{1p}\, V_p}$$

In the reactor:

$$\Delta \rho = 0$$

so that

$$0 = -\Sigma_{2c} - \Sigma_{2p} \frac{m_{2p}}{m_{2c}} \frac{\phi_{2p}}{\phi_{2c}} \frac{V_p}{V_c} - \Sigma_{1c} \frac{m_{1c}}{m_{2c}} \frac{\phi_{1c}}{\phi_{2c}} - \Sigma_{1p} \frac{m_{1p}}{m_{2c}} \frac{\phi_{1p}}{\phi_{2c}} \frac{V_p}{V_c}$$

$$+ p \, \Sigma_{1c} \frac{\phi_{1p}}{\phi_{2c}} + \frac{k_\infty}{p} \Sigma_{2c} \frac{m_{1c}}{m_{2c}}$$

If

$$\frac{\phi_{1c}}{\phi_{2c}} = \frac{k \, \Sigma_{1c}}{p \, \Sigma_{2c}} \left(1 - \frac{p}{k} \frac{\Sigma_{1p}}{\Sigma_{2c}} \frac{m_{1p}}{m_{2c}} \frac{\bar{\phi}_{1p}}{\bar{\phi}_{2c}} \frac{V_p}{V_c} \right)$$

which is

$$\left(\frac{\phi_{1c}}{\phi_{2c}} \right)_\infty \text{ for } \frac{m_{1p}}{m_{2c}} = 1$$

then

$$k_\infty - 1 = \frac{\Sigma_{2p} \, m_{2p} \, \phi_{2p} \, V_p}{\Sigma_{2c} \, m_{2c} \, \phi_{2c} \, V_c} \left(1 + p \frac{\Sigma_{1p}}{\Sigma_{2p}} \frac{m_{1p}}{m_{1c}} \frac{m_{2c}}{m_{2p}} \frac{\phi_{1p}}{\phi_{2p}} \right)$$

If

$$\frac{m_{1c}}{m_{2c}} = p - \frac{\Sigma_{1p} \, m_{1p} \, \phi_{1p} \, V_p}{\Sigma_{1c} \, m_{2c} \, \phi_{1c} \, V_c}$$

which is

$$\left(\frac{m_{1c}}{m_{2c}} \right)_\infty \text{ for } \frac{\phi_{1p}}{\phi_{1c}} = 1$$

then

$$k_\infty - 1 = \frac{\Sigma_{2p}}{\Sigma_{1c}} \left(\frac{m_{2p}}{m_{2c}} \right) \frac{\phi_{2p}}{\phi_{1c}} \frac{V_p}{V_c} \left(1 + \frac{k_\infty}{p} \frac{\Sigma_{2c}\ \Sigma_{1p}\ \phi_{2c}\ \phi_{1p}}{\Sigma_{1c}\ \Sigma_{2p}\ \phi_{1c}\ \phi_{2p}} \right)$$

$\phi_{\frac{1}{v}}$ flux is formulated in the following manner:

The absorption rate in a "$\frac{1}{v}$" absorber is

$$\int_o^\infty \Sigma(E)\ \phi(E)\ dE = \int_o^\infty \Sigma_{\substack{2200 \\ \text{meters/sec}}} \sqrt{\frac{E_o}{E}}\ \phi(E)\ dE$$

where $E_o = 0.0253$ ev

Let us assume a Maxwellian thermal flux and a "$\frac{1}{E}$" fast flux;

$$\phi(E)\ dE = \underbrace{\phi_{th} \frac{E}{kt} \left(\exp \cdot - \frac{E}{kt}\right) \frac{dE}{kt}}_{o \le E \le \infty} + \underbrace{\frac{\phi_r}{E}\ dE}_{E_{RC} \le E \le E_{fission}}$$

thus

$$\phi = \int \phi(E)\ dE = \phi_{th} + \phi_r \ln \frac{E_f}{E_{RC}}$$

Define $\phi_{\frac{1}{v}}$ as follows:

$$\sigma 2200\ \phi_{1/v} = \int_o^\infty \sigma 2200 \sqrt{\frac{E_o}{E}}\ \phi_{th} \frac{E}{kt} \left(\exp \cdot - \frac{E}{kt}\right) \frac{dE}{kt} + \int_{E_{Rc}}^\infty \sigma 2200 \sqrt{\frac{E_o}{E}}\ \frac{\phi_r}{E}\ dE$$

$$\phi_{1/v} = \sqrt{\frac{\pi}{2}} \sqrt{\frac{E_o}{kt}}\ \phi_{th} + 2\sqrt{\frac{E_o}{E_{Rc}}}\ \phi_r$$

Thus, the absorption rate of a "$\frac{1}{v}$" absorber is given by $\sigma_{2200} = \sigma_o$ multiplied by $\phi_{\frac{1}{v}}$.

For materials that are not "$\frac{1}{v}$" absorbers in the thermal region, we can obtain the correct thermal absorption rate by using:

$$\sigma_o f_{1/v} \phi_{1/v} = \sigma_o \frac{\sqrt{2}}{\pi} \frac{f_{1/v}}{F_{th}} \phi_{th} + \sigma_o 2\sqrt{\frac{E_o}{E_{Rc}}} f_{1/v} \phi_r$$

where $f_{1/v}$ is the thermal non-$\frac{1}{v}$ factor. (8)

The epithermal absorptions not accounted for in $f_{\frac{1}{v}} \sigma_o \phi_{1/v}$ must be included in p.

For foils with a thermal disadvantage factor F_{th}, the absorption rate is:

$$\frac{\sigma_o f_{1/v}}{F_{th}} \phi_{1/v} = \sigma_o \frac{\sqrt{\pi}}{2} \frac{f_{1/v}}{F_{th}} \sqrt{\frac{E_o}{kT}} \phi_{th} + \sigma_o 2\sqrt{\frac{E_o}{E_{Rc}}} f_{1/v} \phi_r$$

Again the "excess" epithermal absorptions must go into p.

In the expression for k_∞, the ratio $\frac{\Sigma_{1p} \bar{\phi}_{1p}}{\Sigma_{2p} \bar{\phi}_{2p}}$ occurs when the poison is also a resonance absorber. If the $\phi_{1/v}$ formalism is used, this is the ratio of absorptions in excess of "$\frac{1}{v}$" absorptions.

For $\phi_{1/v} \equiv \sqrt{\frac{\pi}{2}} \phi_{th} + 2\sqrt{\frac{E_o}{E_{Rc}}} \phi_r$

$$\frac{\Sigma_{1p}}{\Sigma_{2p}} \frac{\phi_{1p}}{\phi_{2p}} = \frac{\phi_r \left[|SCI| + |RI| - \frac{f\sigma_o}{F_{th}} 2\sqrt{\frac{E_o}{E_{Rc}}} \right]}{\frac{f\sigma_o}{F_{th}} \left(\frac{\sqrt{\pi}}{2} \phi_{th} + 2\sqrt{\frac{E_o}{E_{Rc}}} \phi_r \right)}$$

$$= \frac{\left[(SCI) + (RI) - \frac{f\sigma_o}{F_{th}} \, 2 \sqrt{\frac{E_o}{E_{Rc}}} \right]}{\frac{f\sigma_o}{F_{th}} \, \phi_{1/v}} \, \beta \, \phi_{th}$$

where (SCI) = sub-cadmium integral

(RI) = resonance integral

$\beta = \frac{\phi_r}{\phi_{th}}$

If the poison is "$\frac{1}{v}$" in the thermal region, $f_{1/v} = 1$. If there are no deviations from "$\frac{1}{v}$" in the epithermal region from E_{Rc} to E_{cc};

$$(SCI) = 2\sigma_o \left[\sqrt{\frac{E_o}{E_{Rc}}} - \sqrt{\frac{E_o}{E_{cc}}} \right]$$

$$B = \frac{\Sigma_{1p} \, \phi_{1p}}{\Sigma_{2p} \, \phi_{2p}} = \left[\frac{(RI) - 2\sigma_o \left[\sqrt{\frac{E_o}{E_{cc}}} + 2\sigma_o \sqrt{\frac{E_o}{E_{Rc}}} \left(1 - \frac{1}{F_{th}} \right) \right]}{\frac{\sigma o}{F_{th}} \, \phi_{1/v}} \right] \beta \, \phi_{th}$$

This factor corresponds to the "B" of the original PCTR theory.(1) (Note that B is incorrectly defined in their paper. For their definition of B, the correction should be $\frac{1}{1 - B}$ instead of $(1 + B)$.) The $(1 + B)$ factor of the old theory is replaced by $(1 + p\frac{m_{1p}}{m_{2p}} \cdot \frac{m_{2c}}{m_{1c}} B)$ in the perturbation theory. (Or for M_p obtained by extrapolation by $(1 + \frac{m_{1p}}{m_{2p}} B)$.)

Measurement of k_∞ for M_p (mass of poison) is carried out as follows:

Let

ρ_c = reactivity of reactor with no poison

ρ_v = reactivity of reactor with void

ρ_p = reactivity of reactor with some poison

Then

$$\Delta\rho_1 = \rho_c - \rho_v = \frac{m_{2c}\,\phi_{2c}\,V_c}{\chi_1}\left[-\Sigma_{2c} - \Sigma_{1c}\left(\frac{m_{1c}}{m_{2c}}\right)\left(\frac{\phi_{1c}}{\phi_{2c}}\right) + p\,\Sigma_{1c}\left(\frac{\phi_{1c}}{\phi_{2c}}\right)_1 + \frac{k_\infty}{p}\,\Sigma_{2c}\left(\frac{m_{1c}}{m_{2c}}\right)_1\right]$$

$$\Delta\rho_2 = \rho_c - \rho_p = \frac{1}{\chi_2}\left[\Sigma_{2p}\left(m_{2p}\,\phi_{2p}\right)_2 V_p + \Sigma_{1p}\left(m_{1p}\phi_{1p}\right)V_p\right]$$

and

$$\frac{\Delta\rho_1}{\Delta\rho_2} = \frac{\Sigma_{2c}\,m_{2c}\,\phi_{2c}\,V_c}{\Sigma_{2p}\,m_{2p}\,\phi_{2p}\,V_p}\left[\frac{-1-\frac{\Sigma_{1c}}{\Sigma_{2c}}\left(\frac{m_{1c}}{m_{2c}}\right)\left(\frac{\phi_{1c}}{\phi_{2c}}\right)+p\,\frac{\Sigma_{1c}}{\Sigma_{2c}}\left(\frac{\phi_{1c}}{\phi_{2c}}\right)+\frac{k}{p}\left(\frac{m_{1c}}{m_{2c}}\right)}{1+\frac{\Sigma_{1p}}{\Sigma_{2p}}\,\frac{m_{1p}}{m_{2p}}\,\frac{\phi_{1p}}{\phi_{2p}}}\right]$$

for $\chi_1 = \chi_2$

Let $\frac{\phi_{1c}}{\phi_{2c}} = \frac{k}{p}\,\frac{\Sigma_{2c}}{\Sigma_{1c}}$, the poisoned infinite flux ratio.

Then

$$k_\infty - 1 = \frac{\Delta\rho_1}{\Delta\rho_2}\;\frac{\Sigma_{2p}\,m_{2p}\,\phi_{2p}\,V_p}{\Sigma_{2c}\,m_{2c}\,\phi_{2c}\,V_c}\left[1+\frac{\Sigma_{1p}}{\Sigma_{2p}}\;\frac{m_{1p}}{m_{2p}}\;\frac{\phi_{1p}}{\phi_{2p}}\right]$$

The following substitutions are made in this formula to obtain the "working" equation for the present experiments.

$$\frac{m_{1p}}{m_{2p}} = \frac{m_{1p}}{m_{1c}}\;\frac{m_{1c}}{m_{2c}}\;\frac{m_{2c}}{m_{2p}}$$

but

$$\frac{m_{1c}}{m_{2c}} = p$$

and by experimental measurement $\frac{m_{2c}}{m_{2p}}$ was found to be equal to 1. 1.

Then $\frac{m_{1p}}{m_{1c}}$ is assumed to be equal to unity and $\frac{m_{2p}}{m_{2c}}$ is assumed to equal to

$$\frac{\phi_{2pp}}{\phi_{2v}}$$

Also

$$B = \frac{\Sigma_{1p}\,\phi_{1p}}{\Sigma_{2p}\,\phi_{2p}}$$

$$\frac{\Sigma_{2p}\,V_p}{\Sigma_{2c}\,V_c} = \frac{M_p \left(\frac{\sigma f}{A}\right)_p}{M_c \left[\sum_i \left(\frac{\sigma f}{A}\right)_i \frac{M_i}{M_c}\right]}$$

Then the "working " equation is

$$k_\infty - 1 = \frac{\Delta\rho_1\, M_p \left(\frac{\sigma f}{A}\right)_p \phi_{2p}\ \phi_{2pp}}{\Delta\rho_2\, M_c \left[\sum_i \left(\frac{\sigma f}{A}\right)_i \frac{M_i}{M_c}\right] \phi_{2c}\ \phi_{2v}} \left[1 + 1.1\ \rho B\right]$$

The error incurred in k_∞ from improper matching is found as follows:

The difference in Σ_{2p} between, (1) that inferred from an experiment in which neither the flux ratio or adjoint ratio is correct and, (2) the correct Σ_{2p}, can be calculated.

Assume

$$\Delta \rho = 0 \text{ but } \frac{\phi'_1}{\phi'_2} \neq \frac{\phi_{1\infty}}{\phi_{2\infty}} \text{ and } \frac{m_1}{m_2} \neq p$$

Now $\Sigma'_2 = \Sigma_2 + \Sigma'_{2p}$

and $-\Sigma'_2 - \Sigma'_{2p} + \frac{k_\infty \Sigma_2}{p} \frac{m_1}{m_2} - \Sigma_1 \frac{m_1}{m_2} \frac{\phi'_1}{\phi'_2} + p\,\Sigma_1 \frac{\phi'_1}{\phi'_2} = 0$

Thus

$$\Sigma'_{2p} - \Sigma_{2p} = \Sigma_1 \left[\frac{m_1}{m_2} \frac{\phi'_1}{\phi'_2} + \frac{\Sigma_2}{\Sigma_1} - \frac{k_\infty}{p} \frac{\Sigma_2}{\Sigma_1} \frac{m_1}{m_2} - p \frac{\phi'_1}{\phi'_2} \frac{\Sigma_{2p}}{\Sigma_1} \right]$$

$$= - \Sigma_1 \left[\frac{\phi'_1}{\phi'_2} - \frac{k_\infty}{p} \quad \frac{\Sigma_2}{\Sigma_1} \right] \left[\frac{m_1}{m_2} - p \right],$$

since

$$\frac{\Sigma_{2p}}{\Sigma_1} = \frac{\Sigma_2}{\Sigma_1} \left(k_\infty - 1 \right).$$

Thus

$$\frac{\Sigma'_{2p}}{\Sigma_2} - \frac{\Sigma_{2p}}{\Sigma_2} = \frac{\left(\frac{\phi'_1}{\phi'_2} - \frac{\phi_{1\infty}}{\phi_{2\infty}} \right) \left(\frac{m_1}{m_2} - \frac{m_{1\infty}}{m_{2\infty}} \right)}{\left(\frac{1}{k_\infty} \right) \left(\frac{k_\infty}{p} \quad \frac{\Sigma_2}{\Sigma_1} \right) \left(p \right)}$$

Thus, if k^*_∞ is the incorrect k_∞

$$k^*_\infty - k_\infty = - \frac{\Sigma'_{2p}}{\Sigma_2} - \frac{\Sigma_{2p}}{\Sigma_2}$$

$$\frac{k^*_\infty - k_\infty}{k_\infty} = - \frac{\left(\frac{\phi'_1}{\phi'_2} - \frac{\phi_{1\infty}}{\phi_{2\infty}}\right)\left(\frac{m_1}{m_2} - \frac{m_{1\infty}}{m_{2\infty}}\right)}{\left(\frac{\phi_{1\infty}}{\phi_{2\infty}}\right)\left(\frac{m_{1p}}{m_{2p}}\right)}$$

or

$$\frac{\delta k_\infty}{k_\infty} = - \frac{\delta\left(\frac{\phi_1}{\phi_2}\right) \quad \delta\left(\frac{m_1}{m_2}\right)}{\left(\frac{\phi_{1\infty}}{\phi_{2\infty}}\right)\left(\frac{m_{1\infty}}{m_{2\infty}}\right)}$$

APPENDIX II

DATA ANALYSIS TABLES AND FORMULAS

The following terms are defined:

γ = fractional U^{235} enrichment

δ = fractional water content $\left(\frac{H_2O}{UO_3 + H_2O}\right)$

M_c = mass of all materials in core tank except the poison

ΔM_t = difference in the aluminum mass of the core tank and void tank

d = density

Σ = macroscopic cross section

Calculation of M_c and Σ_{2c}

$\bar{\gamma}$	Nom H/U	δ	M_c (grams)	$\Sigma_{2c} \times 10^4$
0.01006	4	0.1077	17748	410.2
	5	0.1375	20365	471.8
	6	0.1640	19825	460.3
	7	0.1795	15277	355.1
	8	0.1914	15330	356.8
0.01070	4	0.1064	15786	379.6
	6	0.1554	15895	382.8
	8	0.1836	16356	394.3
0.01159	4	0.1066	15946	404.0
	6	0.1587	16233	410.8
	7	0.1786	16991	429.8
	8	0.1914	14562	368.3

Calculation of M_c, ΔM_t, and d

$\bar{\gamma}$	Nom H/U	V_c (cc)	M_c (gm)	ΔMT (gm)	d gm/cc
0.01006	4	8881	17,748	128	1.998
	5	8881	20,365	128	2.293
	6	8881	19,825	-235	2.232
	7	8881	15,277	- 10	1.720
	8	8881	15,330	128	1.726
0.01070	4	8881	15,786	- 10	1.778
	6	8881	15,895	3	1.790
	8	8881	16,356	3	1.842
0.01159	4	8881	15,946	- 56	1.796
	6	8881	16,233	128	1.828
	7	8881	16,991	- 56	1.913
	8	8881	14,562	128	1.640

Calculation of Hydrogen-to-Uranium Ratio - H/U

$\bar{\gamma}$	Nom H/U	$\bar{\delta}$	H/U
0.01006	4	0.1077	3.772
	5	0.1375	4.999
	6	0.1640	6.164
	7	0.1795	6.881
	8	0.1914	7.449
0.01070	4	0.1064	3.720
	6	0.1554	5.778
	8	0.1836	7.075
0.01159	4	0.1066	3.728
	6	0.1587	5.926
	7	0.1786	6.838
	8	0.1914	7.449

Calculations of $\frac{\overline{m}_p}{\overline{m}_c}$ and $\frac{\overline{\phi}_{pp}}{\overline{\phi}_v}$

The following procedures have been used for obtaining the quantities $\frac{\overline{m}_p}{\overline{m}_c}$ and $\frac{\overline{\phi}_{pp}}{\overline{\phi}_v}$.

1. $\frac{\overline{m}_p}{\overline{m}_c}$ is assumed equal to $\frac{\overline{\phi}_p}{\overline{\phi}_c}$ where

$\overline{\phi}_p$ is the average "$\frac{1}{v}$" flux at the position of the poison before the poison is inserted and $\overline{\phi}_p$ is the average "$\frac{1}{v}$" flux in the cell before posion is added and $\overline{m}_p$ and $\overline{m}_c$ are the similar adjoint fluxes.

2. For the cases in which complete flux traverses are available (i.e. cadmium ratios were taken at many positions), these formulas were used:

$\phi_r' = 2.515\ A_c$ = fast flux or "$\frac{1}{E}$" tail

$\phi_{th}' = 8.295\ A_B - 9.875\ A_c$ = thermal flux or Maxwellian

$\phi_{1/v}' = 7.351\ A_B - 6.533\ A_c$

3. The fluxes at position "P", the position of the poison, are obtained as follows:

ϕ_r' is assumed to be unaffected by the aluminum wall and the copper. It is thus read from the plot of ϕ_r from 2 above.

ϕ_{th}' and $\phi_{1/v}'$ are calculated from this ϕ_r' and bare foil activities.

$\phi_{th}' = 8.295\ A_B - 3.927\ \phi_r'$

$\phi_{1/v} = 7.351\ A_B - 2.598\ \phi_r'$

ϕ_p is the flux at the position "P" with no copper; ϕ_{pp} is the flux at the surface of the copper. (Both at the front to rear center line of the tank.)

Longitudinal traverses showed that the average flux

$$\bar{\phi}_p = 0.992 \ \phi_p$$

and

$$\bar{\phi}_{pp} = \frac{\phi_{pp}}{F_{cu}} = 0.982 \ \phi_{pp}$$

where F_{cu} = disadvantage factor for 20 mil cu.

4. ϕ_v, the "$\frac{1}{v}$" flux in the void, is obtained from CR measurements. The average of positions "C" and "E" is used in each case. (Center and edge positions.)

For the cases in which ϕ_v was not measured, estimates were used. For cases with a six-inch buffer, the "average" of all six-inch buffer cases was used.

5. For the longitudinal traverse the data for three positions include A_c's and A_b's. ϕ_r, ϕ_{th}, $\phi_{1/v}$ were calculated from these and a curve for ϕ_r was drawn. For the other positions for which A_b were known, ϕ_r was read from the curve and ϕ_{th} and $\phi_{1/v}$ calculated from ϕ_r and A_b.

6. The average radial flux $\bar{\phi}_R$ is calculated from the $\phi_{1/v}$ curves for the core tanks.

$$\bar{\phi}_R = \frac{\sum_i R_i \, \phi_i \, \Delta R_i}{\sum_i R_i \, \Delta R_i}$$

Since the foils were each one-half inch long and measured the average flux along a half-inch portion of the radius $\Delta R_i = \frac{1}{2}$" and

$$\overline{\phi}_R = \frac{\sum_i R_i \phi_i}{\sum_i R_i}$$

where R_i = radius to center of foil

and $\phi_i = \phi_{1/v}$ at that radius, read from the curve.

The reason for using the values from the curve instead of the values of the points was that the curves are drawn with the fact that the normalization of the points is artificial. Thus the curves will often lie below the points, weighting the "E" position heavily. The error introduced by such a procedure cannot be evaluated, but the averages appear to be good to better than 0.5 per cent because the flux is not varying rapidly in almost all cases.

7. For the cases in which only the "C" and "E" center and edge position data are available the curves were drawn with the general shape suggested by the complete traverse cases. This was a subjective procedure; again, the error introduced was small, because of the small variation in flux in the core tanks.

8. $\overline{\phi}_c$ was obtained from $\overline{\phi}_R$ and $\overline{\phi}_v$

$$\overline{\phi}_c = \overline{\phi}_R \ \overline{\phi}_v$$

Working Formulas for ϕ_r; ϕ_{th}; $\phi_{1/v}$; β ; B_{cu}

In terms of resonance integrals of gold (5 mil)

A_B = activity of a 5 mil gold foil irradiated bare

A'_B = A_B normalized like ϕ_r

A_C = activity of the 5 mil gold foil irradiated with a 40-mil cadmium cover.

$$\phi_r = \frac{1}{2}\sqrt{\frac{E_{cc}}{E_o}}\;A_c = 2.515\;A_c$$

$$\phi_{th} = \left[\frac{(RI)}{2}\sqrt{\frac{E_{cc}}{E_o}}\;\frac{2}{\sqrt{\pi}}\;\frac{F}{\sigma f}\right]A_B - \left[\frac{(RI)+(SCI)}{2}\sqrt{\frac{E_{cc}}{E_o}}\;\frac{2}{\sqrt{\pi}}\;\frac{F}{\sigma f}\right]A_c$$

$$= 8.295\;A_B - 9.875\;A_c = 8.295\;A'_B - 3.927\;\phi_r$$

$$\phi_{1/v} = \left[\frac{(RI)}{2}\sqrt{\frac{E_{cc}}{E_o}}\;\frac{F}{\sigma f}\right]A_B - \left[\frac{(RI)+(SCI)}{2}\sqrt{\frac{E_{cc}}{E_o}}\;\frac{F}{\sigma f} - \sqrt{\frac{E_{cc}}{E_{Rc}}}\right]A_c$$

$$= 7.351\;A_B - 6.533\;A_c = 7.351\;A'_B - 2.598\;\phi_r$$

$$\beta = \frac{\phi_r}{\phi_{th}} = \frac{\frac{f\,\bar{\sigma}}{F}\;\frac{\sqrt{\pi}}{2}\;\frac{1}{(RI)}}{\frac{A_B}{A_c} - \left[\frac{(RI)+(SCI)}{(RI)}\right]} = \frac{0.3032}{\frac{A_B}{A_c} - 1.191}$$

Working Formulas for ξ and (RI) (5 mil Au)

$$\xi = \frac{\left(\frac{A_B}{A_c}\right)_{1/v} - 1}{\left(\frac{A_B}{A_c}\right)_{Au} - 1} = \left[\frac{1 + 0.5469\;\beta}{1 + 0.6285\;\beta}\right]\left[0.02709\,(RI)\right]$$

$$(RI) = \left[2\sqrt{\frac{E_o}{E_{cc}}}\;\frac{\sigma f}{F}\right]\left[\frac{1 + \frac{2}{\sqrt{\pi}}\;\frac{F}{\sigma f}\;(SCI)\;\beta}{1 + \frac{4}{\sqrt{\pi}}\left(\sqrt{\frac{E_o}{E_{Rc}}} - \sqrt{\frac{E_o}{E_{cc}}}\right)\beta}\right]\xi$$

APPENDIX III

METHODS AND FORMULAS FOR CORRECTIONS TO $\Delta k'_\infty$

A. Determination of $\Delta k'_\infty$ which corresponds to the flux that is characteristic of the system (mixture plus Al cans but no Cu).

The usual technique that was employed here was to plot Δk_∞ versus CR_c (cadmium ratio in core) and to use interpolation to find the value of $\Delta k'_\infty$ corresponding to the correct flux ratio, and to find the errors of this value of $\Delta k'_\infty$. In these experiments, this cannot be done for all cases because the fastest and slowest reactor loadings would not cause much change in CR_c or CR_e (cadmium ratio at edge); this was because the six-inch buffers effectively brought the flux ratios to near the correct one. Consequently, for those cases in which the two values of CR_c were close together, the best that could be done was (1) to average the two values of $\Delta k'_\infty$ to get an estimate of the correct value, or (2) to use that value corresponding to the flux ratio nearest the correct one.

The following procedure was used:

1. Interpolation was used for all cases that were amenable to this technique.
2. For other cases in which the correct CR lies between the two values of CR, the average of the two values of Δk_∞ was used.
3. For other cases in which the correct CR lies outside at the two values of CR, the nearest value of Δk_∞ was used as the correct one.

A typical k_∞ versus cadmium ratio plot is shown in Figure 15 in which the correct $\Delta k'_\infty$ was found by interpolation of the data.

B. Correction due to difference in masses of tanks.

As shown in the theoretical two-group treatment

$$\Delta \rho_{12} = \frac{1}{\chi} \Sigma_{2c} m_{2c} \phi_{2c} V_c \left[k_\infty - 1 + \frac{\left(\Sigma_{2av} V - \Sigma_{2ac} V\right) m_{2a} \phi_{2a}}{\Sigma_{2c} V_c m_{2c} \phi_{2c}} \right]$$

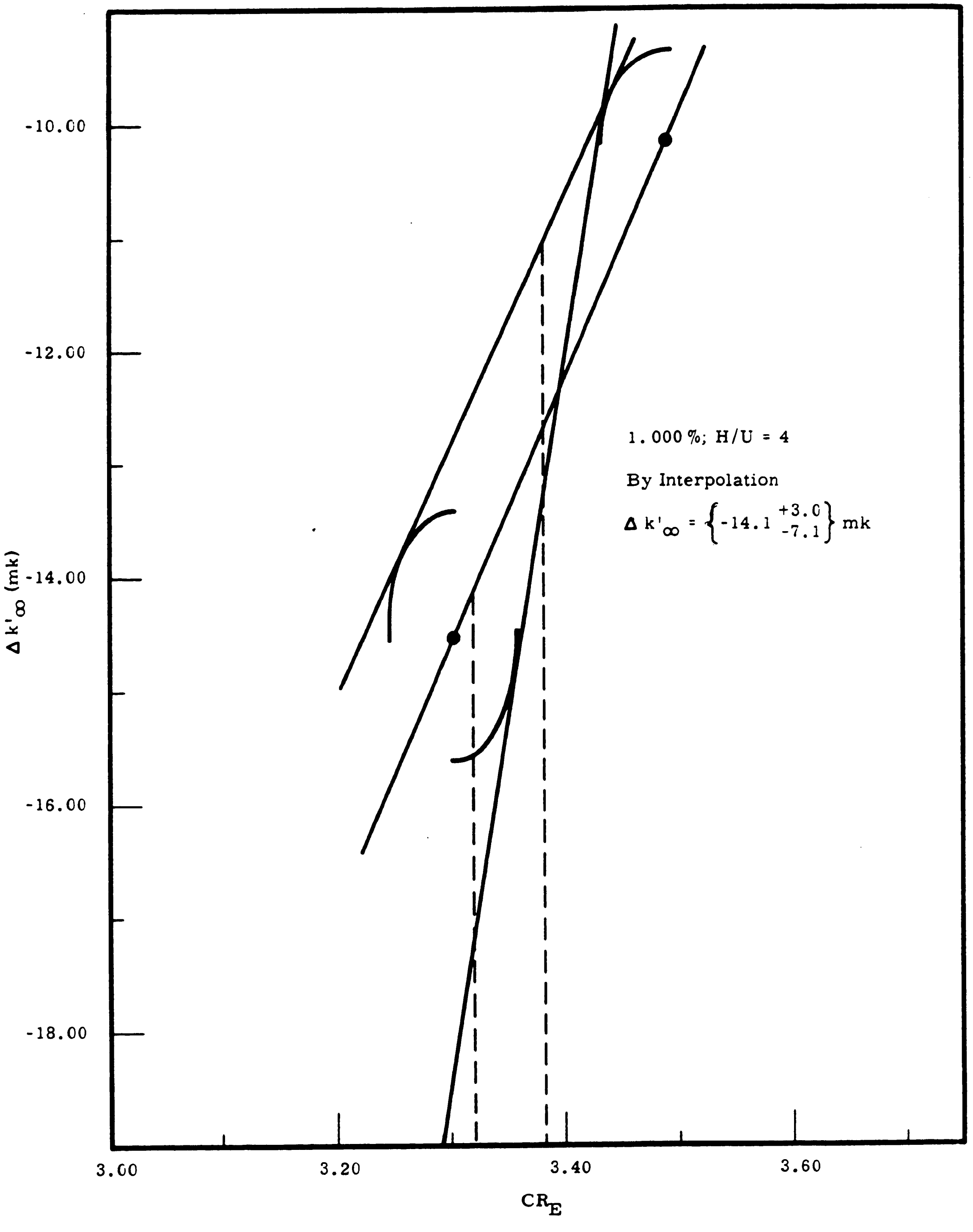


FIGURE 15

Plot of k_∞ versus Cadmium Ratio

so that

$$(k_\infty - 1)_{\text{corrected}} = (k_\infty - 1)_{\text{uncorrected}} - \frac{\Delta(\Sigma_2 V)_a}{\Sigma_{2c} V_c} \frac{m_{2a} \phi_{2a}}{m_{2c} \phi_{2c}}$$

$$\simeq (k_\infty - 1)_{\text{uncorrected}} + \frac{\left|M_{ac} - M_{av}\right| \left|\frac{\sigma 2}{A}\right|_A}{M_c \left|\frac{\sigma}{A}\right|_c}$$

where

Mac = mass of aluminum in core tank
Mav = mass of aluminum in void tank

C. Estimate of Effect of Nitrogen in Mixtures on Σ_{2c}

Nitrogen comes in two forms: (1) bound in nitrates
(2) free in voids

Let δ_N = mass fraction of $NO_3 = \frac{M_{NO_3}}{M}$

Then $\frac{M_{NO_3} \left|\frac{\sigma}{A}\right| NO_3}{M} = \delta_N \left|\frac{\sigma}{A}\right|_{NO_3}$

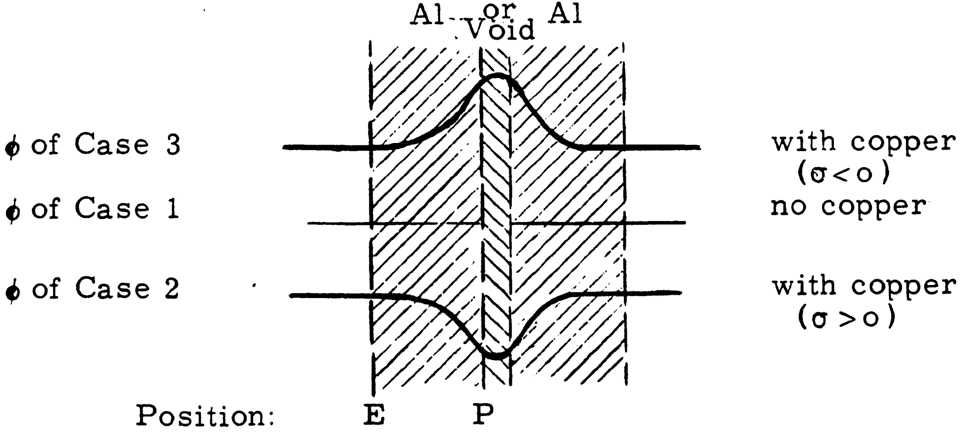
This item was then calculated from measured nitrate concentrations in the system.

The packed theoretical density of dry UO_3 is 7.3 m/c. With water attached, the density will decrease to about 5 m/cc. The densities obtained during the experiment were 1.6 - 2.3 m/cc.

Thus, about $\frac{3.6}{5.0} \lesssim 0.7$ of the tanks were void. The correction for free nitrate was calculated from this.

D. Calculation of Absorptions in Containment Vessels

Let us assume that any change in A (total number of absorptions) was due to the placing of Cu strips around the cell. (If A is to be reduced, these strips must be considered as sources of thermal neutrons that are proportional to the flux at the Cu.) An alternate method is to consider the negative absorptions in copper as being strips of neutron producing material (J metal) as is done in evaluating $k_\infty < 1$ in the PCTR. (9)



As approximation, assume flux is about flat when no copper is present.

Notation: ϕ_{1E}; ϕ_{2E}; ϕ_{3E} are thermal fluxes at position E for cases 1, 2, 3 respectively

ϕ_p, ϕ_{pp2}, ϕ_{pp3} are thermal fluxes at position P for cases 1, 2, 3 respectively.

λ is fraction of circumference of can covered by copper

M^*_{Cu} is mass of Cu that covers all of can (ends excluded)

The ratios $\frac{\phi_{pp2}}{\phi_{E2}}$, $\frac{\phi_{pp3}}{\phi_{E3}}$, $\frac{\phi_p}{\phi_{E1}} \simeq 1$ are independent of the amount of Cu present since strips are used. Also, $\frac{\phi_{E2}}{\phi_{E1}}$ and $\frac{\phi_{E3}}{\phi_{E1}}$ are constants.

(Effects at edges of strips are neglected.)

Case 1: No copper on can.

$$A_1 = \Sigma_{Al} V_{Al} \bar{\phi}_{Al} \simeq \Sigma_{Al} V_{Al} \phi_{E1}$$

Case 2: Some copper ($\sigma_a > 0$) on can.

$$A_+ = A^{(Al)} + A^{(Cu)}$$

$$\simeq (1 - \lambda) \Sigma_{Al} V_{Al} \phi_{E1} + \lambda \Sigma_{Al} V_{Al} \left(\frac{\phi_{E2} + \phi_{pp2}}{2} \right)$$

$$+ \lambda \frac{\Sigma_{cu} V_{cu} \phi_{pp2}}{F_{cu}}$$

$$\simeq A_1 \left[1 - \lambda + \frac{\lambda}{2} \left(\frac{\phi_{E2}}{\phi_{E1}} + \frac{\phi_{pp2}}{\phi_{E1}} \right) + \lambda \frac{\Sigma_{cu} V_{cu} \phi_{pp2}}{\Sigma_{Al} V_{Al} \phi_{Al} F_{cu}} \right]$$

Case 3: Some copper ($\sigma_a < 0$) on can.

$$A_- = A_3^{(Al)} + A_3^{(Cu)}$$

$$\simeq (1 - \lambda) \Sigma_{Al} V_{Al} \phi_{E1} + \lambda \Sigma_{Al} V_{Al} \left(\frac{\phi_{E3} + \phi_{pp3}}{2} \right)$$

$$- \lambda \Sigma_{cu} V_{cu} \phi_{pp3} F_{cu} .$$

$$\simeq A_1 \left[1 - \lambda + \frac{\lambda}{2} \left(\frac{\phi_{E3}}{\phi_{E1}} + \frac{\phi_{pp3}}{\phi_{E1}} \right) - \lambda \frac{\Sigma_{cu} V_{cu} \phi_{pp3} F_{cu}}{\Sigma_{Al} V_{Al} \phi_{E1}} \right]$$

When sufficient - copper is placed on the can, the net absorptions in Al + Cu will be zero. At this condition, the flux ratio in the UO_3 will be that of the UO_3 mixture alone.

When $A = 0$, $\lambda = \lambda_o$

$$\lambda_o = \frac{1}{1 - \frac{\phi_{E3}}{2\,\phi_{E1}}\left(1 + \frac{\phi_{pp3}}{\phi_{E3}}\left[1 - \frac{2\,F_{cu}\,\Sigma_{cu}\,V_{cu}}{\Sigma_{Al}\,V_{Al}}\right]\right)}$$

The flux ratio for the correctly "poisoned" cell will be obtained when some -copper is removed for cases for $k_\infty > 1$ or when some additional - copper is placed on can for $k_\infty < 1$. Since this is a small amount of - copper compared to $(-\lambda_o\, M^*_{Cu})$ for our cases, we can use the absorptions required to poison the cell to $k_\infty = 1$ in the wrong flux as a good approximation.

Neglecting the epithermal absorptions in the copper:

$$A_{cu_\infty} \simeq (k_\infty - 1)\ \Sigma_{2c}\ \bar{\phi}_{2c}\ V_c\ \frac{\phi_{2v}}{\phi_{2p}}$$

$$M_{cu\,\infty} \simeq (k_\infty - 1)\ \Sigma_{2c}\ \bar{\phi}_{2c}\ V_c\left(\frac{\phi_{2v}}{\phi_{2p}}\right)$$

$$\lambda_\infty = \lambda_o - \lambda'$$

$$\cong \lambda_o - \frac{M_{cup}}{M} = \lambda_o - \frac{\left(k_\infty - 1\right)\ M_c\ \left[\frac{1}{M}\sum_i\left(\frac{\sigma f}{A}\right)\right]\left(\phi_{2c}\right)\ \left(\phi_{2v}\right)}{F_{cu}\ \left(\frac{\sigma f}{A}\right)_{cu}\ M_{cu}\ \left(\phi_{2p}\right)\ \left(\phi_{pp2}\right)}$$

To include epithermal abs in copper divide by (1 + 1.1 pB)

$$\lambda_\infty \cong \lambda_o - \frac{(k_\infty - 1)\ M_c \left[\frac{1}{M_c} \sum_i \left(\frac{\sigma f}{A}\right)_i\right] \left(\bar{\phi}_{2c}\right) \left(\bar{\phi}_{2v}\right)}{F_{cu}\ M^*_{cu} \left(\frac{\sigma f}{A}\right)_{cu} \left(\phi_{2p}\right) \left(\phi_{2pp}\right)}$$

$$\cong \lambda_o - \left(\frac{\Delta \rho_{cv}}{\Delta \rho_{cp}}\right)' \left(\frac{M_{cu}}{M^*_{cu\,r}}\right)$$

where M_{Cu} = actual mass of copper used in experiment as poison in reactivity measurements and $\frac{\Delta \rho}{\Delta \rho}$ corresponds to the M_{Cu} for $\Delta k'_\infty$.

DETAILS OF ERROR ANALYSIS

A. Calculation of $\sigma\left(\frac{\rho_c - \rho_v}{\rho_c - \rho_p}\right)$

$$\sigma\left(\frac{\rho_c - \rho_v}{\rho_c - \rho_p}\right) = \frac{\sqrt{\left(\rho_c - \rho_v\right)^2 \sigma^2\left(\rho_p\right) + \left(\rho_c - \rho_p\right)^2 \sigma^2\left(\rho_v\right) + \left(\rho_v - \rho_p\right)^2 \sigma^2\left(\rho_c\right)}}{\left(\rho_c - \rho_p\right)^2}$$

For $\sigma(\rho_c) = \sigma(\rho_r) = \sigma(\rho_p) = \sigma(\rho)$

$$\sigma\left(\frac{\rho_c - \rho_r}{\rho_c - \rho_p}\right) = \frac{\sqrt{\left(\rho_c - \rho_v\right)^2 + \left(\rho_c - \rho_p\right)^2 + \left(\rho_v - \rho_p\right)^2}}{\left(\rho_c - \rho_p\right)^2} \sigma(\rho)$$

For cases when two independent values of ρ_c are available:

$$\sigma\left(\frac{\rho_c - \rho_v}{\rho_c - \rho_p}\right) = \frac{\sqrt{2\left[\left(\rho_{c1} - \rho_v\right)^2 + \left(\rho_{c2} - \rho_v\right)^2\right]}}{\left(\rho_{c2} - \rho_p\right)^2} \sigma(\rho)$$

B. Calculation of σ(CR)

When a cadmium ratio (CR) is known from one set of irradiations, the standard deviation is obtained from

$\sigma(\mathrm{CR}) = 2f(N)(\mathrm{CR})^*$

$f(N)$ = fractional deviation for one foil

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* Neglects uncertainty due to masses of foils which is small for this experiment.

Calculation of $\sigma(CR)$ when Two Measurements of the Cadmium-Covered Foil Activity are Available

$$CR = \frac{A_B}{\bar{A}_C}$$

$$\sigma^2(CR) = (CR)^2 \left[2f^2(N) + \frac{1}{2} f^2 \ (N) \left(\frac{A_{C1}^2 + A_{C2}^2}{\bar{A}_C^2} \right) \right]$$

$$\simeq 3f^2 \ (N) \ (CR)^2$$

$$\sigma(CR) \simeq \sqrt{3} \ f \ (N) \ (CR)$$

C. Calculation of $\sigma(\bar{\phi} V)$

$$\bar{\phi}_V = \frac{\phi_{1/v}(C) + \phi_{1/v}(E)}{2}$$

$$= \frac{7.35}{2} \left[A_B(C) + A_B(E) \right] + \frac{6.533}{2} \left[A_C(C) + A_C(E) \right]$$

for He tank (C) = center and (E) = edge

but

$$A_B(C) = \frac{N_B(C)}{M_B} \text{ and } A_B(E) = \frac{N_B(E)}{M_B}$$

where N_B = unnormalized counting rate of bare foil.

where M_B = counting rate of monitor for bare foils,

and similarly for $A_C(C)$ and $A_C(E)$

$$\text{Thus } \bar{\phi}_v = \frac{7.251}{2M_B}\left[N_B(E) + N_B(C)\right] + \frac{6.533}{2M_C}\left[N_C(E) + N_C(C)\right]$$

$$\text{Thus } \sigma^2\left(\bar{\phi}_v\right) \cong \left(\frac{7.251}{2M_B}\right)^2 \left[\sigma^2\left(N_B(E)\right) + \sigma^2\left(N_B(C)\right) + \frac{\left[N_B(E) + N_B(C)\right]^2}{M_B^2}\sigma^2(M_B)\right] + \left(\frac{6.533}{M_B}\right)^2 \left[\sigma^2\left(N_C(E)\right) + \sigma^2\left(N_C(C)\right) + \frac{\left[N_C(E) + N_C(C)\right]^2}{M_C^2}\sigma^2\left(M_C\right)\right]$$

D. Calculation of $\sigma^2\left[\sum_i \left(\frac{\sigma f}{A}\right)_i \frac{M_i}{M_C}\right]$ Due to $\bar{\gamma}$ and $\bar{\delta}$ Only

Considering the errors introduced by uncertainties in enrichment and water analysis into the quantity of $\sum_i \left(\frac{\sigma f}{A}\right)_i \frac{M_i}{M_C}$ yields the formula:

$$\sigma^2\left[\sum_i \left(\frac{\sigma f}{A}\right)_i \frac{M_i}{M_C}\right] \simeq \left(1 - \bar{\delta}\right)^2 \frac{\left[\left(\frac{\sigma f}{A}\right)_{235} - \left(\frac{\sigma f}{A}\right)_{238}\right]^2}{\Omega^2}\sigma^2(\bar{\gamma}) + \left[\left(\frac{\sigma f}{A}\right)_{H_2O} - \frac{\bar{\gamma}\left(\frac{\sigma f}{A}\right)_{235} + \left(1 - \bar{\gamma}\right)\left(\frac{\sigma f}{A}\right)_{238}}{\Omega}\right]^2 \sigma^2(\bar{\delta})$$

E. Errors in Cadmium Ratio Analysis

To obtain the "correct" CR, a linear extrapolation or interpolation was used. A confidence-limit circle is drawn about each of the two known points and tangents to these circles determine the error in the correct CR.

Not all cases are amenable to this technique because the error may be infinite. In such cases, an average of the measured cadmium ratios must be used as the correct CR and its error computed so as to include the two measured values.

$$\overline{CR} = \frac{1}{2} (CR_1 + CR_2) \qquad \text{(Both CR measured at same position)}$$

$$\sigma^2 (\overline{CR}) = \frac{1}{4} \left[\sigma^2 (CR_1) + \sigma^2 (CR_2) \right]$$

or

$$\sigma^2 (\overline{CR}) = (\overline{CR} - CR_1)^2 + (\overline{CR} - CR_2)^2$$

whichever is larger.

In the cases which interpolation is possible, the radius of the circle for a given loading was given by

$$r = \sigma (CR) \left[2 \ln (1 - \sqrt{p}) ^{-1} \right]^{\frac{1}{2}}$$

Where p is the desired confidence interval and σ (CR) is the σ of the cadmium ratios (σ is assumed to be the same for the two CR's).

APPENDIX V

TABLE OF CONSTANTS FOR k_∞ CALCULATIONS

	U^{235}	U^{238}	H	O	H_2O
A	235.117	238.125	1.00827	16.00435	18.02089
$f_{1/v}$	0.981	1.00	1.00	1.00	-
σ_a	694 ± 8	2.75 ± 0.04	0.332 ± 0.002	< 0.0002	0.660 ± 0.006
σ_f	582 ± 6	-	-	-	-
$1 + \alpha$	1.19 ± 0.01	-	-	-	-
ν	2.47 ± 0.03	-	-	-	-
σ_s	10 ± 2	8.3 ± 0.02	20.0 ± 0.2	3.8 ± 0.3	43.8 ± 0.4
ξ	0.00849	0.00838	1.000	0.120	0.9237
$\frac{\sigma_a f_{1/v}}{A}$	2.896	0.01155	0.3293	0.00001249	0.03662
$\frac{\sigma_f f_{1/v}}{A}$	2.428	-	-	-	-
$\frac{\sigma_s}{A}$	0.04253	0.03486	19.84	0.2374	2.431
$\frac{\xi \sigma_s}{A}$	0.0003611	0.0002921	19.84	0.02849	2.245

where

σ's are from BNL-325

A's are physical scale

$$\xi = 1 + \frac{(A - 1)^2}{2A} \ln \left(\frac{A - 1}{A + 1}\right)$$

$$\sigma_{H_2O} = 2\sigma_H + \sigma_o$$

$$\xi_{H_2O} = \frac{2\xi_H \sigma_H + \xi_o \sigma_o}{\sigma_{H_2O}}$$

ACKNOWLEDGEMENTS

The authors wish to acknowledge the contributions of M. D. Alford, Chemical Research and Development Operation, for his assistance in the preparation, sampling, loading, and unloading of the experimental vessels; J. J. Woody who performed many of the machine calculations necessary to analyze the experimental data, and the personnel of the Experimental Reactors Operation for their assistance in the operation of the PCTR.

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